Review Topics from Chapter 3&4

- Fourier Series
- Fourier Transform
- Linear Time Invariant (LTI) Systems
- Energy-Type Signals
- Power-Type Signals

Fourier Series Representation for Periodic Signals

Definition: Let the signal $x(t)$ be a periodic signal with period T. $x(t)$ can be expanded in terms of the complex exponential signals

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt}}$$

$$\boxed{X_n = \frac{1}{T}\int_T x(t) e^{-j\omega_0 nt} dt}$$

- The coefficients X_n are called the *Fourier series coefficients* of the signal $x(t)$. These are, in general, complex numbers.

- $\omega_0 = 2\pi/T$ [rad] is called the *fundamental frequency*. The frequencies of the complex exponential signals are multiples of this frequency.

Alternative Representations for *Real* Periodic Signals

$$X_n = \frac{1}{T}\int_T x(t) e^{-j\omega_0 nt} dt = \underbrace{\frac{1}{T}\int_T x(t)\cos(\omega_0 nt) dt}_{A_n} - j\underbrace{\frac{1}{T}\int_T x(t)\sin(\omega_0 nt) dt}_{B_n}$$

$$X_n = A_n - jB_n \longrightarrow \left|X_n\right| = \sqrt{A_n^2 + B_n^2} \qquad \theta_n = -arctg\left(B_n / A_n\right)$$

$$x(t) = X_0 + 2\sum_{n=1}^{\infty}\left|X_n\right|\cos(\omega_0 nt + \theta_n)$$

where $X_0 = \frac{1}{T}\int_T x(t) dt$

Prove it!

Main steps in the proof :

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} = X_0 + \sum_{n=1}^{\infty}\left(X_n e^{j\omega_0 nt} + X_{-n} e^{-j\omega_0 nt}\right) = X_0 + 2\sum_{n=1}^{\infty}\mathrm{Re}\left(X_n e^{j\omega_0 nt}\right)$$

Alternative Representations for *Real* Periodic Signals (Cont'd)

$$X_n = \frac{1}{T}\int_T x(t)e^{-j\omega_0 nt}dt = \underbrace{\frac{1}{T}\int_T x(t)\cos(\omega_0 nt)dt}_{A_n} - j\underbrace{\frac{1}{T}\int_T x(t)\sin(\omega_0 nt)dt}_{B_n}$$

$$\boxed{x(t) = X_0 + 2\sum_{n=1}^{\infty}\left[A_n\cos(\omega_0 nt) + B_n\sin(\omega_0 nt)\right]}$$

Prove it!

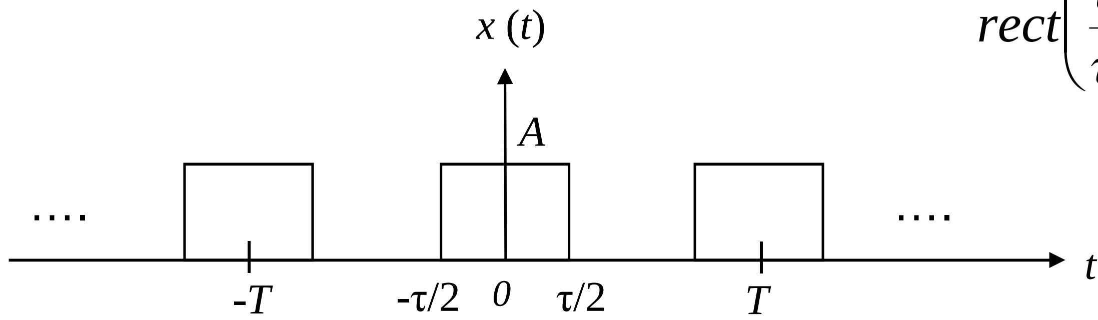
<u>Main steps in the proof</u> :

$$x(t) = X_0 + 2\sum_{n=1}^{\infty}\mathrm{Re}\left(X_n e^{j\omega_0 nt}\right) = X_0 + 2\sum_{n=1}^{\infty}\mathrm{Re}\{(A_n - jB_n)[\cos(\omega_0 nt) + j\sin(\omega_0 nt)]\}$$

$$\left.\begin{array}{l}\text{If } x(t) \text{ is an even function, i.e. } x(t)=x(-t),\ B_n=0.\\ \text{If } x(t) \text{ is an odd function, i.e. } x(t)=-x(-t),\ A_n=0.\end{array}\right\}\ \text{Prove them!}$$

Example: Periodic Gate Function

$$rect\left(\frac{t}{\tau}\right) \overset{def}{=} \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$



$x(t)$ is an even periodic function. $x(t)=x(-t)$

$$X_n = \frac{1}{T}\int_T x(t)\cos(\omega_0 n t)dt = \frac{A}{T}\int_{-\tau/2}^{\tau/2}\cos(\omega_0 n t)dt = \frac{A}{T}\frac{2}{\omega_0 n}\sin\left(\omega_0 n\frac{\tau}{2}\right)$$

For n=0, the above gives $\frac{0}{0}$

$$X_0 = \frac{1}{T}\int_T x(t)dt = \frac{A}{T}\int_{-\tau/2}^{\tau/2}dt = \frac{A\tau}{T}$$

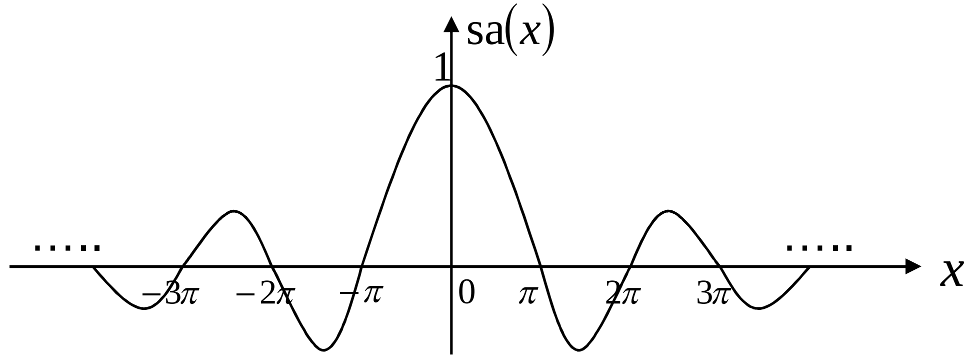
Alternatively, starting from general expression and using L'Hospital rule:

$$X_0 = \lim_{n\to 0}\frac{2A}{T}\frac{\sin\left(\omega_0 n\frac{\tau}{2}\right)}{\omega_0 n} = \lim_{n\to 0}\frac{2A}{T}\frac{\left[\sin\left(\omega_0 n\frac{\tau}{2}\right)\right]'}{[\omega_0 n]'} = \lim_{n\to 0}\frac{A}{T}\tau\cos\left(\omega_0 n\frac{\tau}{2}\right) = \frac{A}{T}\tau$$

Differentiation w.r.t. n

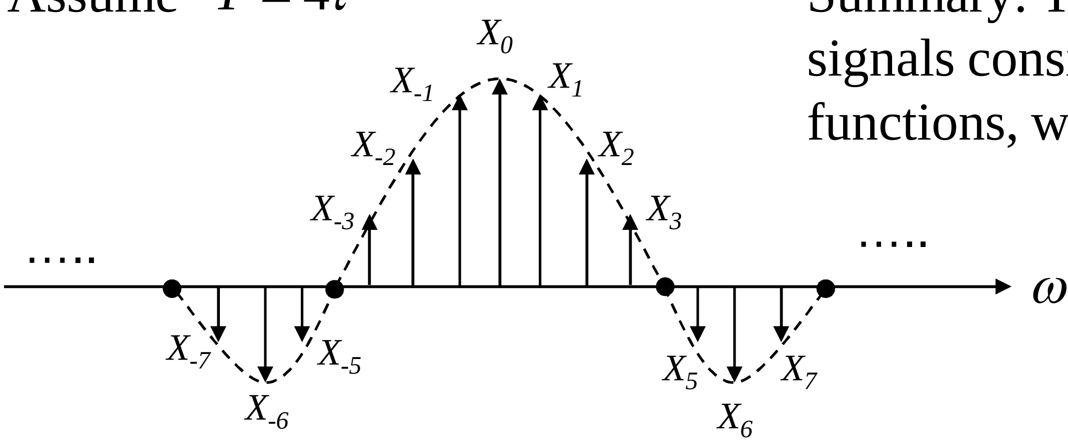
$$\mathrm{sa}(x) \stackrel{def}{=} \frac{\sin(x)}{x}$$

“**s**ine over **a**rgument ” function



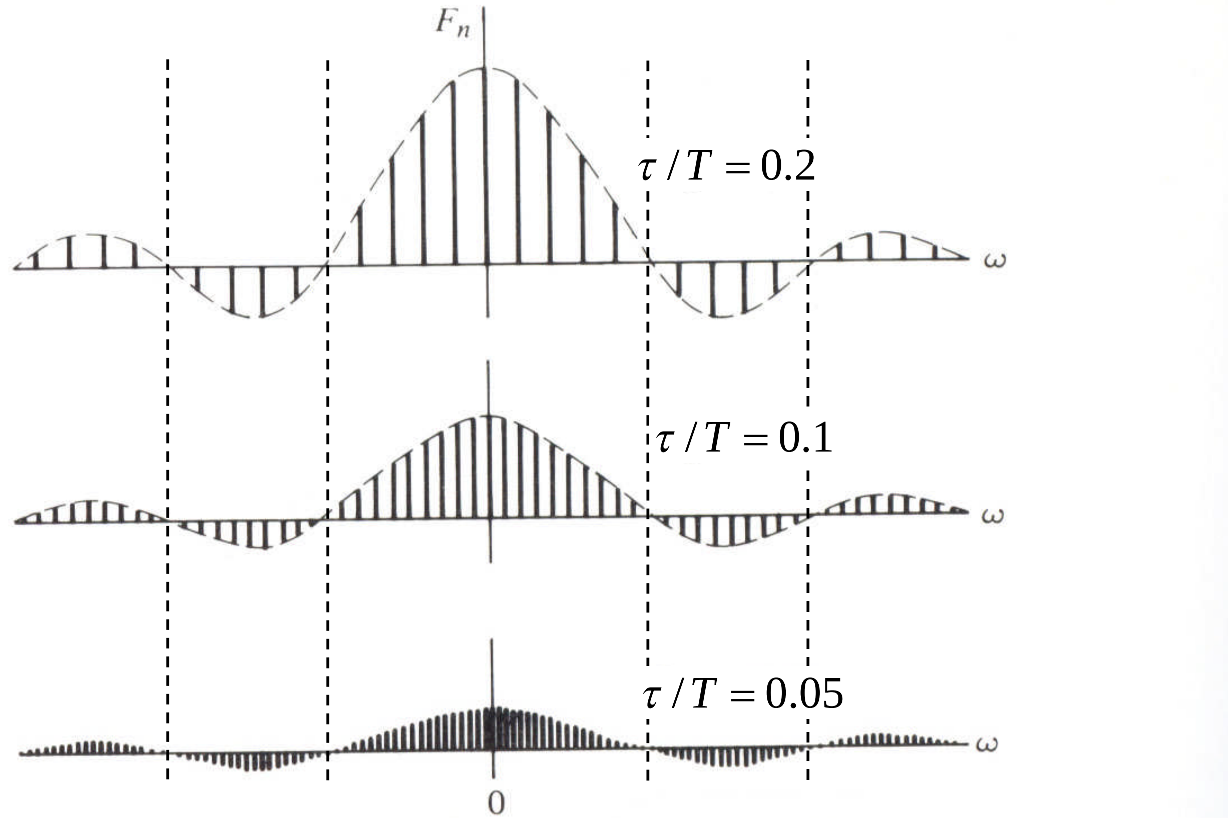
$$X_n = \frac{A}{T}\frac{2}{\omega_0 n}\sin\left(\omega_0 n \frac{\tau}{2}\right) = \frac{A\tau}{T}\mathrm{sa}\left(\frac{\omega_0 n \tau}{2}\right)$$

Assume $T = 4\tau$



Summary: The spectrum of periodic signals consists of a series of impulse functions, which is a discrete spectrum.

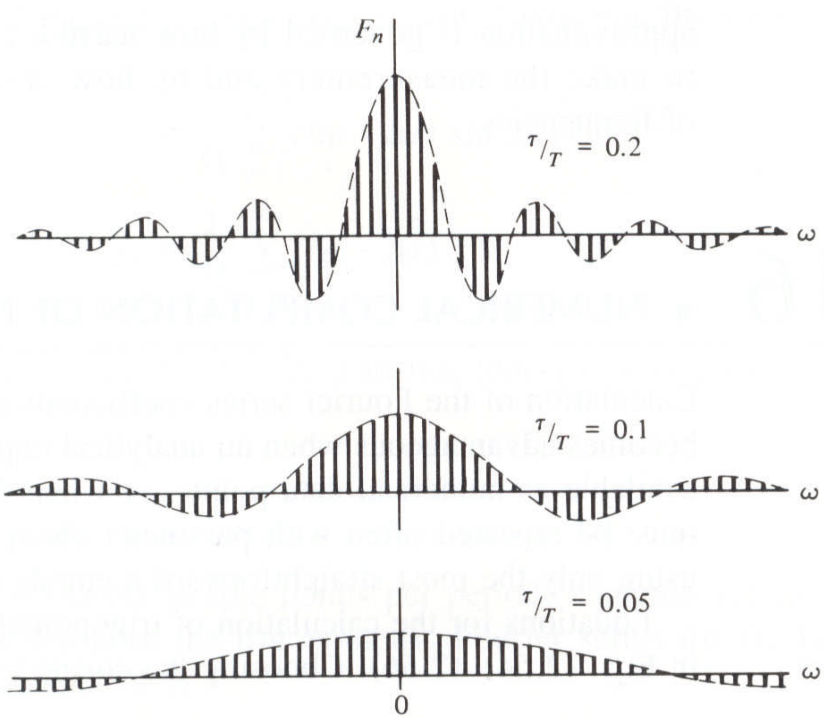
Spectrum for various values of τ/T, --- τ fixed

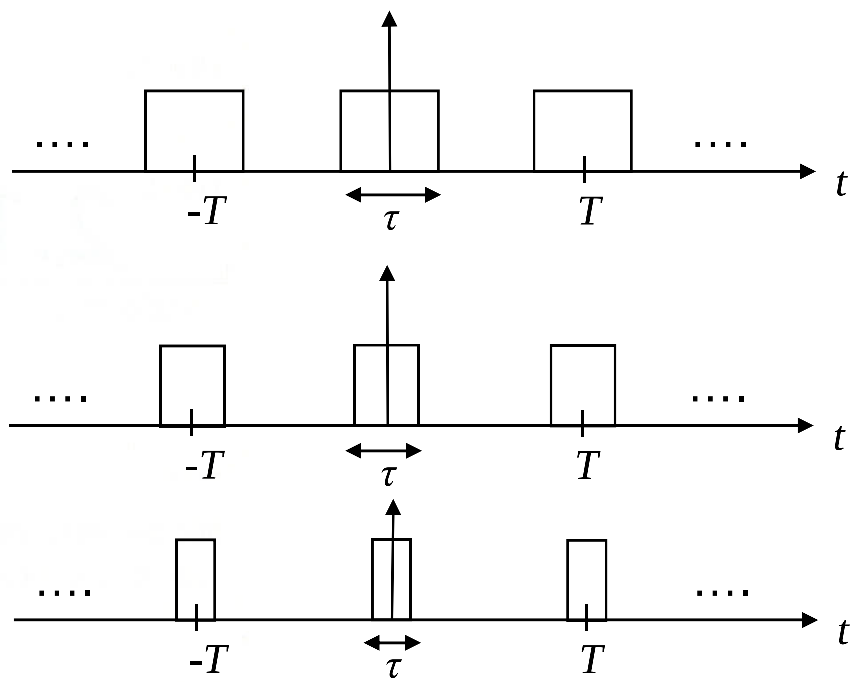


Observations:

- The amplitude is proportional to $1/T$.
- The spacing between lines is proportional to $1/T$.
- Zero crossings of the spectrum remain the same. (not dependent on T).

Spectrum for various values of τ/T, --- T fixed





Observations:

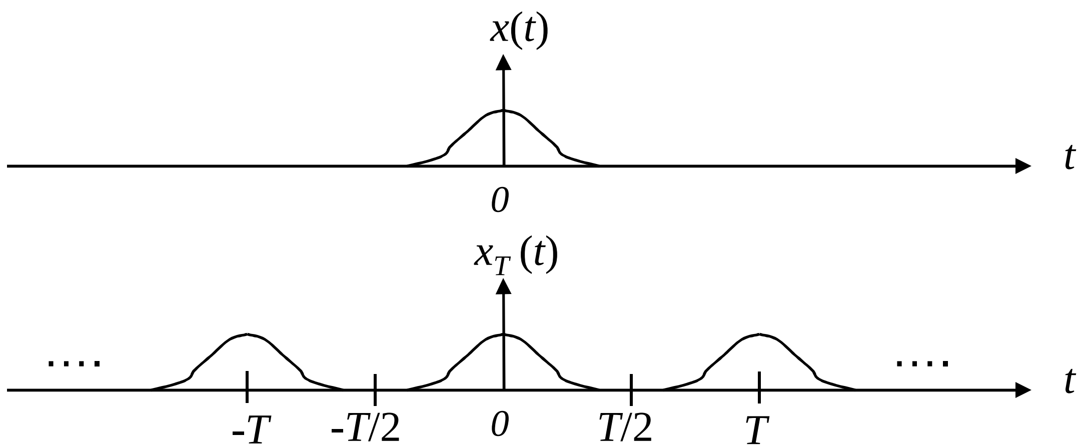
- The amplitude is proportional to τ.

- The spectrum spreads as τ decreases. → <u>There is an inverse relationship between pulse width in time (τ) and frequency *spread* of the spectrum.</u>

Fourier Transform

The Fourier series is a means for expanding a periodic signal in terms of complex exponentials. Now, consider an *aperiodic* function. How can we express it as a sum of exponential signals?

Construct a new periodic function $x_T(t)$ based on the original aperiodic signal $x(t)$.



$$\lim_{T \to \infty} x_T(t) = x(t)$$

$x_T(t)$ is a periodic signal and can be represented by Fourier series.

$$x_T(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} \quad \text{where} \quad X_n = \frac{1}{T}\int_T x_T(t) e^{-j\omega_0 nt} dt$$

$$\omega_n \overset{def}{=} n\omega_0$$

$$X(\omega_n) \overset{def}{=} TX_n$$

$$x_T(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} X(\omega_n) e^{j\omega_n t} \qquad X(\omega_n) = \int_T x_T(t) e^{-j\omega_n t} dt$$

Defining $\Delta\omega \overset{def}{=} 2\pi/T$ $\longrightarrow$ $$x_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} X(\omega_n) e^{j\omega_n t}\Delta\omega$$

As T becomes large,
$\Delta\omega$ becomes smaller.

$$\lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} X(\omega_n) e^{j\omega_n t}\Delta\omega$$

Riemann Integral

In the limiting case, the discrete lines in the spectrum merge and frequency spectrum becomes continuous.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform $$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier Transform $$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$X(\omega)$ is also known as *spectral-density function* of $x(t)$.

$$\boxed{x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)}$$ Fourier Transform Pair

Relationship between Fourier Series & Fourier Transform

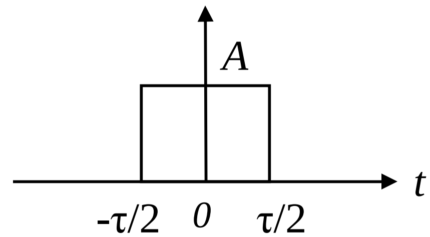
$$X_n = \frac{1}{T} X(\omega)\Big|_{\omega \to n\omega_0}$$

$$X(\omega) = T\, X_n\Big|_{\omega_0 \to \frac{\omega}{n}}$$

Example: Gate function

$$rect\left(\frac{t}{\tau}\right) \stackrel{def}{=} \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$x(t) = A\ rect(t/\tau)$



$x(t)$ is an even aperiodic function.

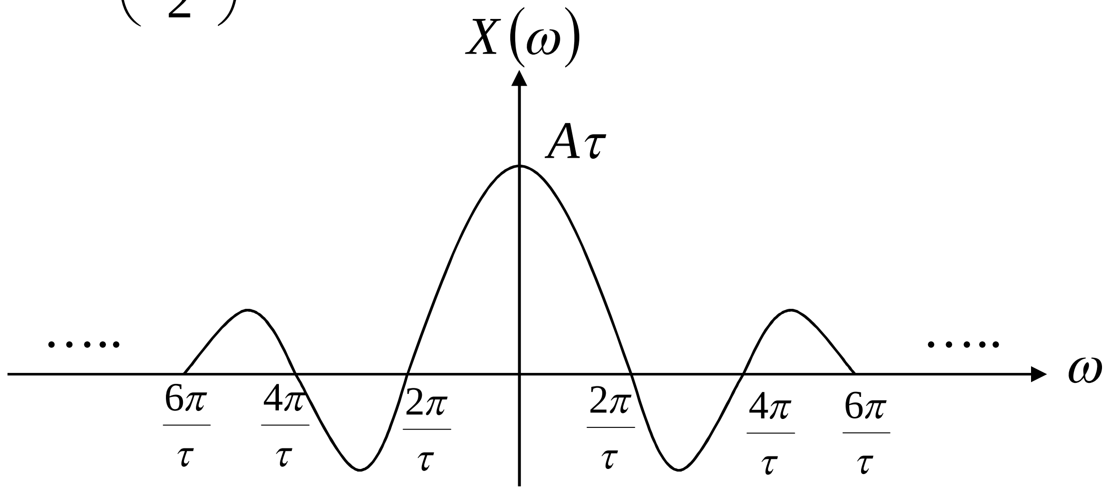
$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt \; = A\int_{-\tau/2}^{\tau/2} \cos(\omega t)dt \; = \frac{A}{\omega}\sin(\omega t)\Big|_{t=-\tau/2}^{t=\tau/2}$$

$$= \frac{A}{\omega} 2\sin\left(\omega\frac{\tau}{2}\right) \; = A\tau \;\; \text{sa}\left(\frac{\omega\tau}{2}\right)$$

<u>Relation to Fourier Series</u>: We had obtained Fourier series for a periodic gate function (see previous example).

$$X(\omega) = T\, X_n\Big|_{\omega_0 \to \frac{\omega}{n}} = T\,\frac{A\tau}{T}\text{sa}\left(\frac{\omega_0 n\tau}{2}\right)\Bigg|_{\omega_0 \to \frac{\omega}{n}} \; = A\tau \;\; \text{sa}\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \ \mathrm{sa}\left(\frac{\omega\tau}{2}\right)$$



Observation: Compare the spectrum to periodic signals. Aperiodic signals have continuous spectrum.

Fourier Transform for Periodic Signals

A periodic signal can be represented by its exponential Fourier Series, i.e.

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} \qquad \omega_0 = 2\pi/T$$

Taking the Fourier transform,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt}\right\}$$

$$\mathcal{F}\{x(t)\} = \sum_{n=-\infty}^{+\infty} X_n \mathcal{F}\left(e^{j\omega_0 nt}\right) = 2\pi \sum_{n=-\infty}^{+\infty} X_n \delta(\omega - n\omega_0)$$

The Fourier transform of a periodic signal consists of a set of impulses located at the multiples of the fundamental frequency. The weight of each impulse is 2π times the value of its corresponding coefficient in the Fourier series.

Properties of the Fourier Transform

- **Linearity (Superposition)**

$$\left.\begin{aligned} x_1(t) &\leftrightarrow X_1(\omega) \\ x_2(t) &\leftrightarrow X_2(\omega) \end{aligned}\right\} a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

a_1, a_2: arbitrary constants

Proof:

$$\mathcal{F}\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{+\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-j\omega t} dt$$

Linearity of integration

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\omega t} dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\omega t} dt$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

- **Complex Conjugate**

$$x(t) \leftrightarrow X(\omega)$$

$x^*(t)$: Complex conjugate of $x(t)$

$$x^*(t) \leftrightarrow X^*(-\omega)$$

Proof:

$$\mathcal{F}\left\{x^*(t)\right\} = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\omega t} dt \quad = \left[\int_{-\infty}^{+\infty} x(t)\, e^{j\omega t} dt\right]^* = X^*(-\omega)$$

- **Duality**

$$x(t) \leftrightarrow X(\omega) \qquad \mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$

$$\mathcal{F}\{X(-t)\} = 2\pi x(\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Let $u = -\omega$ $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(-u) e^{-jut} du$$

Let $t = \omega$ $$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(-u) e^{-ju\omega} du$$

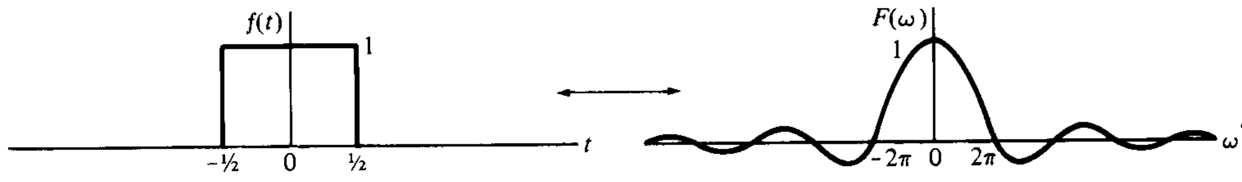
Substituting t for u $$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(-t) e^{-j\omega t} dt$$

These substitutions correspond to a change between t and ω

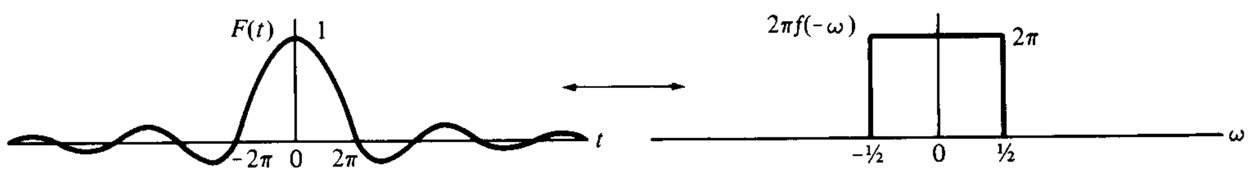
$$2\pi x(\omega) = \mathcal{F}\{X(-t)\}$$

Example: Gate function

$$rect(t) \leftrightarrow sa\left(\frac{\omega}{2}\right)$$



Using duality property $\quad sa\left(\frac{t}{2}\right) \leftrightarrow 2\pi \; rect(-\omega)$



$$2\pi \; rect(-\omega) = 2\pi \; rect(\omega)$$

↓

Even function

- **Coordinate Scaling**

$$x(t) \leftrightarrow X(\omega) \qquad x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

Proof:

$$\mathrm{F}\{x(\alpha t)\} = \int_{-\infty}^{+\infty} x(\alpha t)\, e^{-j\omega t} dt$$

Let $u = \alpha t \longrightarrow t = u/\alpha \quad dt = du/\alpha$

$$\mathrm{F}\{x(\alpha t)\} = \begin{cases} \dfrac{1}{\alpha} \displaystyle\int_{-\infty}^{+\infty} x(\alpha t)\, e^{-j\omega \frac{u}{\alpha}} du, & \alpha > 0 \\ -\dfrac{1}{\alpha} \displaystyle\int_{-\infty}^{+\infty} x(\alpha t)\, e^{-j\omega \frac{u}{\alpha}} du, & \alpha < 0 \end{cases} = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

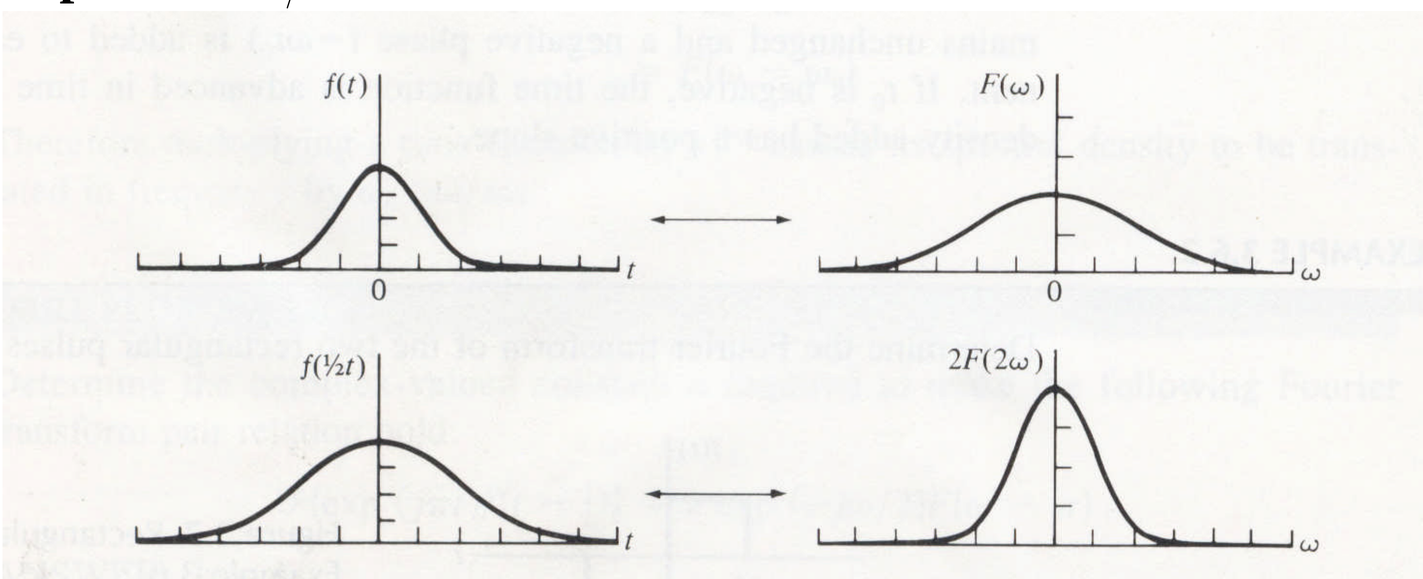
$0 < \alpha < 1$ $\quad x(\alpha t)$:expanded version of $x(t)$

$\mathcal{F}\{x(\alpha t)\}$:compressed version of $x(t)$

$\alpha > 1$ $\quad x(\alpha t)$:compressed version of $x(t)$

$\mathcal{F}\{x(\alpha t)\}$:expanded version of $x(t)$

Example: $\alpha = 1/2$



- **Time Shifting**

$$x(t) \leftrightarrow X(\omega) \qquad x(t \mp t_0) \leftrightarrow e^{\mp j\omega t_0} X(\omega)$$

Proof:

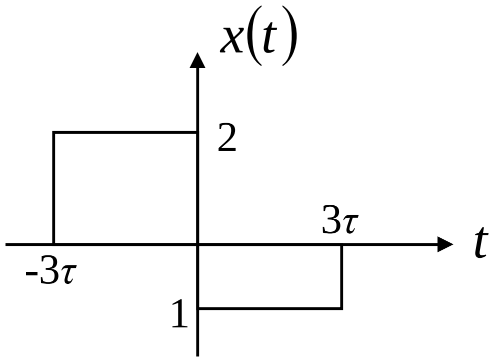
$$\mathrm{F}\,\{x(t \mp t_0)\} = \int_{-\infty}^{+\infty} x(t \mp t_0)\; e^{-j\omega t} dt$$

$$\text{Let } u = t \mp t_0 \quad \longrightarrow \quad t = u \pm t_0 \qquad dt = du$$

$$\mathrm{F}\,\{x(t \mp t_0)\} = \int_{-\infty}^{+\infty} x(u)\; e^{-j\omega(u \pm t_0)} du$$

$$= e^{\mp j\omega t_0} \int_{-\infty}^{+\infty} x(u)\; e^{-j\omega u} du \; = e^{\mp j\omega t_0} X(\omega)$$

Example: Find the Fourier transform of $x(t)$



$$x(t) = 2\mathrm{rect}\left(\frac{t+3\tau/2}{3\tau}\right) - \mathrm{rect}\left(\frac{t-3\tau/2}{3\tau}\right)$$

$$\mathcal{F}\{\mathrm{rect}(t/\tau)\} = \tau \ \mathrm{sa}(\omega\tau/2)$$

Using scaling and time shifting properties,

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{t \mp 3\tau/2}{3\tau}\right)\right\} = 3\tau \ \mathrm{sa}\left(\frac{3\omega\tau}{2}\right) e^{\mp j\omega\frac{3}{2}\tau}$$

Using linearity,

$$\mathcal{F}\{x(t)\} = 6\tau \ \mathrm{sa}\left(\frac{3\omega\tau}{2}\right) e^{+j\omega\frac{3}{2}\tau} - 3\tau \ \mathrm{sa}\left(\frac{3\omega\tau}{2}\right) e^{-j\omega\frac{3}{2}\tau}$$

- **Frequency Shifting**

$$x(t) \leftrightarrow X(\omega) \qquad\qquad e^{\mp j\omega_0 t} x(t) \leftrightarrow X(\omega \pm \omega_0)$$

Proof:

$$\mathcal{F}\left\{e^{\mp j\omega_0 t} x(t)\right\} = \int_{-\infty}^{+\infty} x(t)\; e^{\mp j\omega_0 t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(t)\; e^{-j(\omega \pm \omega_0)t} dt$$

$$= X(\omega \pm \omega_0)$$

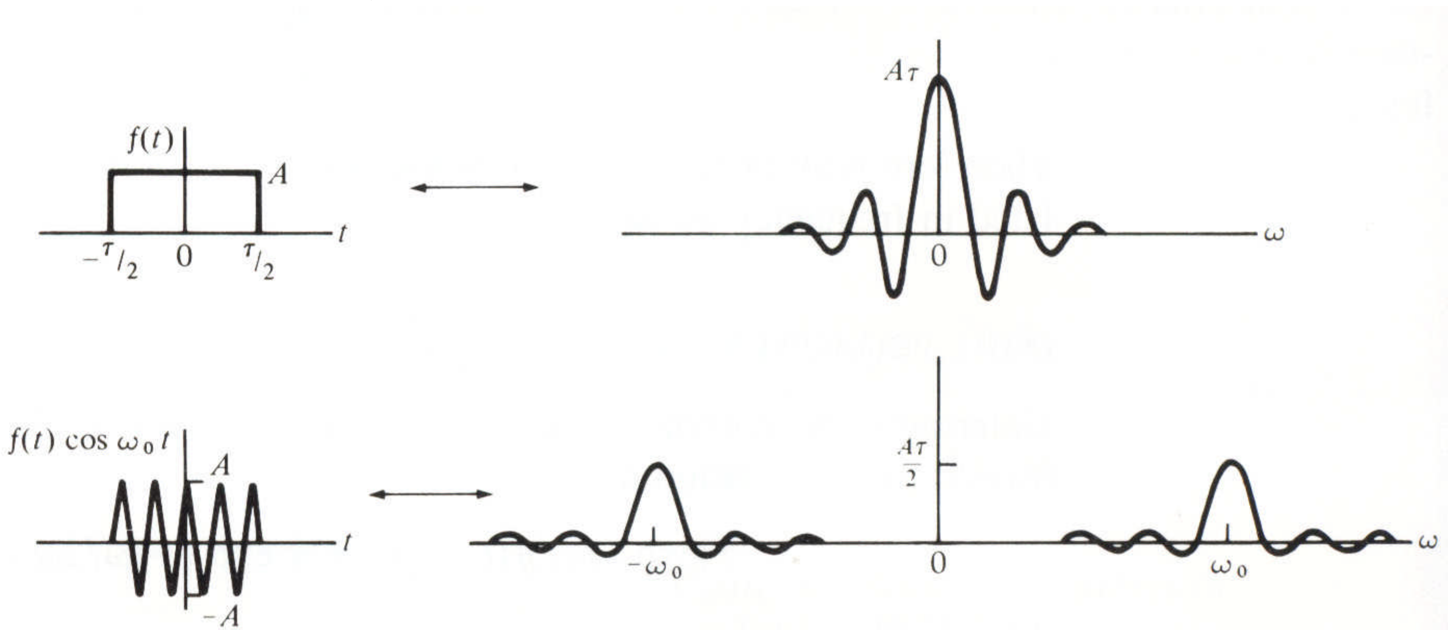
Example 5: Find the Fourier transform of $Arect(t/\tau)\cos(\omega_0 t)$

$$f(t) = A\ rect\left(\frac{t}{\tau}\right) \qquad F(\omega) = A\tau\ \mathrm{sa}\left(\frac{\omega\tau}{2}\right)$$

$$f(t)\cos\omega_0 t = \frac{1}{2} f(t) e^{-j\omega_0 t} + \frac{1}{2} f(t) e^{+j\omega_0 t}$$

Using frequency shifting properties,

$$\mathcal{F}\left\{A\ rect\left(\frac{t}{\tau}\right)\cos(\omega_0 t)\right\} = \frac{A\tau}{2} sa\left[\frac{(\omega+\omega_0)\tau}{2}\right] + \frac{A\tau}{2} sa\left[\frac{(\omega-\omega_0)\tau}{2}\right]$$



- **Convolution**

$$\left.\begin{array}{l} x_1(t) \leftrightarrow X_1(\omega) \\ x_2(t) \leftrightarrow X_2(\omega) \end{array}\right\} \quad x_1(t) \otimes x_2(t) \leftrightarrow X_1(\omega) X_2(\omega)$$

Proof:

$$\mathcal{F}\{x_1(t) \otimes x_2(t)\} = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-j\omega t} dt$$

Changing the order of integration

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left[\int_{-\infty}^{+\infty} x_2(t-\tau) e^{-j\omega t} dt \right] d\tau$$

Time-shift property

$$= \int_{-\infty}^{+\infty} x_1(\tau) e^{-j\tau\omega} X_2(\omega) d\tau$$

$$= X_2(\omega) \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-j\tau\omega} d\tau \quad = X_1(\omega) X_2(\omega)$$

- **Differentiation**

$$x(t) \leftrightarrow X(\omega) \qquad \frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left[X(\omega)e^{j\omega t}\right]d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \mathrm{F}^{-1}\{j\omega X(\omega)\} \quad \longrightarrow \quad \mathrm{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega)$$

- **Integration**

$$x(t) \leftrightarrow X(\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

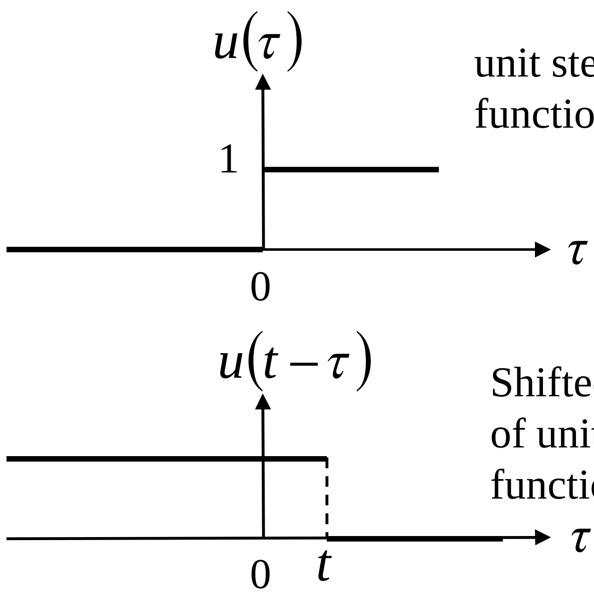
Proof:

$$x(t) \otimes u(t) = \int_{-\infty}^{+\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

$$\mathcal{F}\{x(t) \otimes u(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\}$$

$$X(\omega)U(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$ Prove it!



unit step function

Shifted version of unit step function

Linear Time Invariant (LTI) Systems

LTI systems provide good and accurate models for a large class of communication systems. Some basic components of transmitters and receivers, such as filters, amplifiers and equalizers can be also modeled as LTI systems.

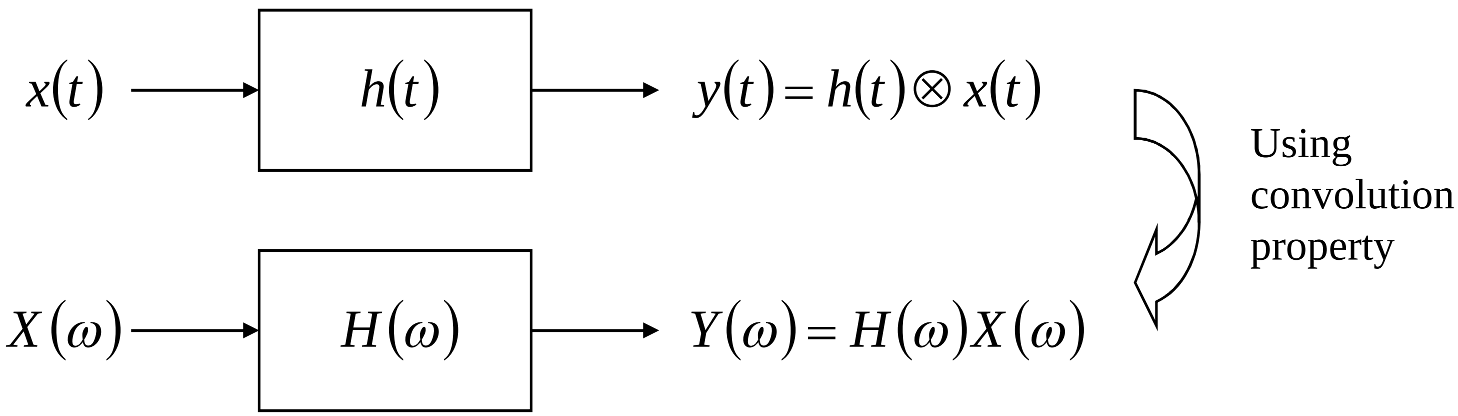
The *impulse response* $h(t)$ of a system is the system's response to a unit impulse.

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

$$\begin{aligned} y(t) &= h(t) \otimes x(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \end{aligned}$$

The *frequency transfer function* is the Fourier Transform of the impulse response.

$$H(\omega) = \mathcal{F}\{h(t)\}$$



In time domain, we have the convolution integral. On the other hand, in the frequency domain, the input-output relation is much simpler, just given as a multiplication.

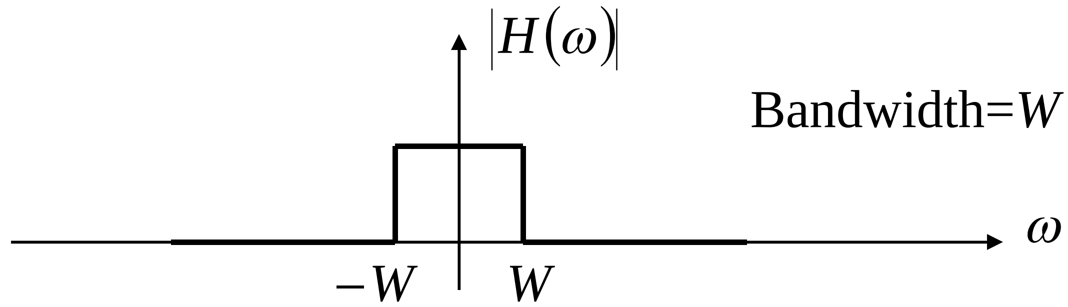
An LTI acts as a filter on the various frequency components of the input signal. It might affect both its amplitude and phase.

$$|Y(\omega)|e^{j\theta_y(\omega)} = |H(\omega)|e^{j\theta_h(\omega)}|X(\omega)|e^{j\theta_x(\omega)}$$

$$|Y(\omega)| = |H(\omega)|\ |X(\omega)|$$

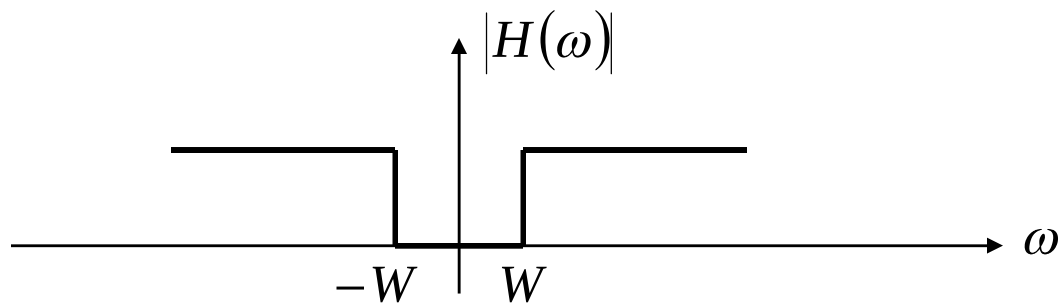
$$\theta_y(\omega) = \theta_h(\omega) + \theta_x(\omega)$$

Ideal Low Pass
Filter (LPF)

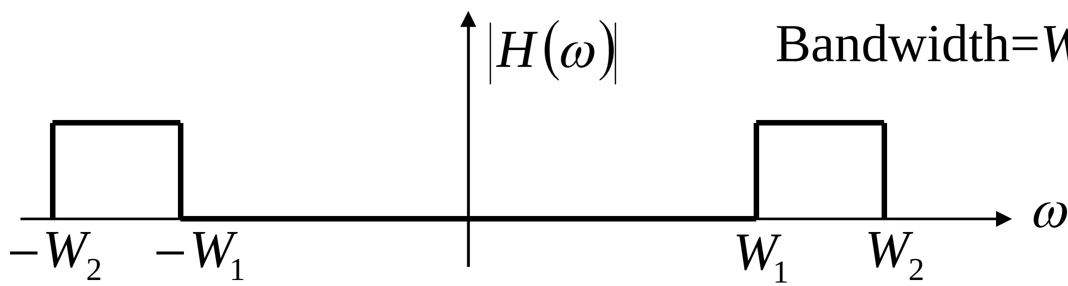


Bandwidth=W [rad]

Ideal High Pass
Filter (HPF)



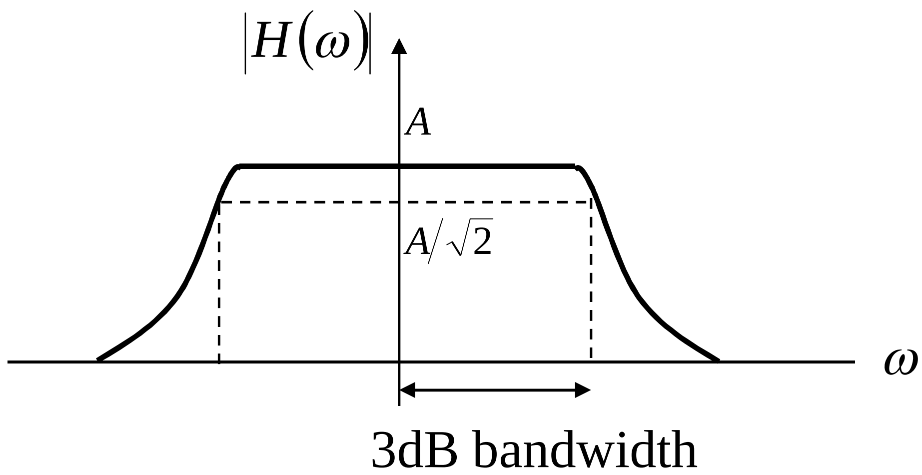
Ideal Band Pass
Filter (BPF)



Bandwidth=W_1-W_2

In practice, filters do not have sharp transitions as ideal ones. The bandwidth is then defined as " the interval of positive frequencies over which the magnitude of transfer function remains within a given numerical factor ". Although different criterion might be used, a widely used definition is *3dB bandwidth*.

According to this criteria, the bandwidth is defined as the band of frequencies at which the magnitude of the transfer function is at least $1/\sqrt{2}$ of its max. value. It is called as *3dB bandwith*, because reducing the amplitude by a factor of $\sqrt{2}$ corresponds to a decrease of 3dB on logarithmic scale.



Energy Signals

Definition: The energy of a signal $x(t)$ is defined by

$$E_x \overset{def}{=} \int_{-\infty}^{+\infty} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Definition: The signal $x(t)$ is *energy-type signal (*or *energy signal)* if E_x is finite.

Example:

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} A^2 \cos^2(\omega_0 t + \theta) dt = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{A^2}{2} [1 + \cos(2\omega_0 t + 2\theta)] dt$$

$$= \lim_{T\to\infty} \frac{A^2 T}{2} + \frac{A^2}{4\omega_0} \sin(2\omega_0 t + 2\theta) \Big|_{-T/2}^{T/2} = \infty$$

Not an energy signal!

Parseval's Theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

$|X(\omega)|^2$: Energy spectral density

Proof:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt$$

Replace by Inverse Fourier Transform

$$= \int_{-\infty}^{+\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

Change the order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega) \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

Use the definition of Fourier Transform

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\; X(\omega)\; d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

Power Signals

Some signals have infinite energy, but they may have a finite time-average of energy. This time-average of energy is called *average power.*

Definition: The average power of a signal $x(t)$ is given by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Definition: The signal $x(t)$ is *power-type signal* (or power signal) if $0 < P_x < \infty$

A signal can not be both power- and energy-type, because for energy-type signals $P_x = 0$ and for power-type signals $E_x = \infty$

A signal may be neither energy-type nor power-type.

Example: $x(t) = A\cos(\omega_0 t + \theta)$

$E_x \to \infty$ (See previous example)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(\omega_0 t + \theta) dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} [1 + \cos(2\omega_0 t + 2\theta)] dt$$

$$= \lim_{T\to\infty} \frac{A^2 T}{2T} + \frac{A^2}{4\omega_0 T} \sin(2\omega_0 t + 2\theta)\Big|_{-T/2}^{T/2} dt = \frac{A^2}{2} < \infty$$ Power signal

Example: $x(t) = e^{-at} u(t)$ $a > 0$

$$E_x = \int_0^\infty e^{-2at} dt = \frac{1}{2a} < \infty$$ Energy signal

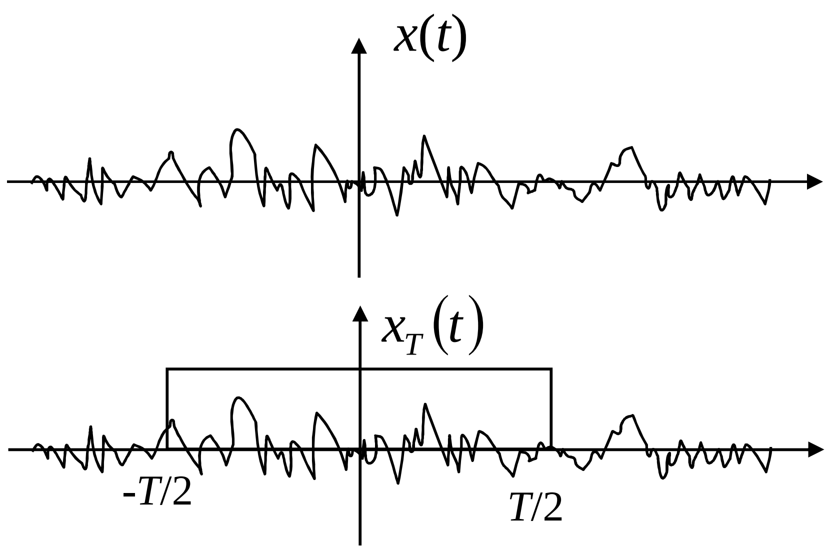
$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} e^{-2at} dt = \lim_{T\to\infty} \frac{1}{2aT} \left(1 - e^{-aT}\right) = 0$$

Power Spectral Density

In an analogy with the energy spectral density, we can define *power spectral density* (PSD) for power signals.

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \overset{def}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) d\omega$$

Derivation of $S_x(\omega)$



$x(t)$: power signal

$x_T(t)$: truncated to –T/2 and T/2

$$x_T(t) = x(t)\text{rect}(t/T)$$

$$\int_{-T/2}^{+T/2} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

Using Parseval's Theorem

Taking limit of both sides

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

Using PSD definition

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) d\omega = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

This relationship should hold over each frequency increment.

Define the *cumulative power spectrum* (i.e. cumulative amount of power for all frequency components below a given frequency ω)

$$G_x(\omega) \overset{def}{=} \frac{1}{2\pi}\int_{-\infty}^{\omega} S_x(u)du = \lim_{T\to\infty}\frac{1}{T}\frac{1}{2\pi}\int_{-\infty}^{\omega}\left|X_T(u)\right|^2 du$$

If an interchange in the order of limiting operation and the integration is valid,

$$2\pi G_x(\omega) = \int_{-\infty}^{\omega} S_x(u)du = \int_{-\infty}^{\omega}\lim_{T\to\infty}\frac{\left|X_T(u)\right|^2}{T}du$$

If $G_x(\omega)$ is differentiable,

$$2\pi\frac{d}{d\omega}G_x(\omega) = S_x(\omega)$$

Under the above assumptions

$$\boxed{S_x(\omega) = \lim_{T\to\infty}\frac{\left|X_T(\omega)\right|^2}{T}}$$

Power Spectral Density for Periodic Signals

The previous discussion on PSD holds for any general power signal. Now, assume a periodic power signal

For a periodic signal, each period contains a replica of the function and the limiting operation can be omitted as long as *T* is taken as the period.

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)\,dt$$

Express in terms of Fourier series

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{m=-\infty}^{\infty} X_m e^{jm\omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right)dt$$

Changing the order of summation and integration

$$= \sum_{m=-\infty}^{\infty} X_m \sum_{n=-\infty}^{\infty} X_n^* \frac{1}{T}\int_{-T/2}^{+T/2} e^{j(m-n)\omega_0 t}\,dt$$

$$\frac{1}{T}\int_{-T/2}^{+T/2} e^{j(m-n)\omega_0 t}\,dt = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Fourier Transform $X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega - n\omega_0)$

Power Spectral Density $S_x(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(\omega - n\omega_0)$

PSD of periodic signals have discrete components. It consists of a series of impulse functions with weights corresponding to the magnitude squared of respective Fourier series coefficients.

$$P_x = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Periodic Function

Aperiodic Function

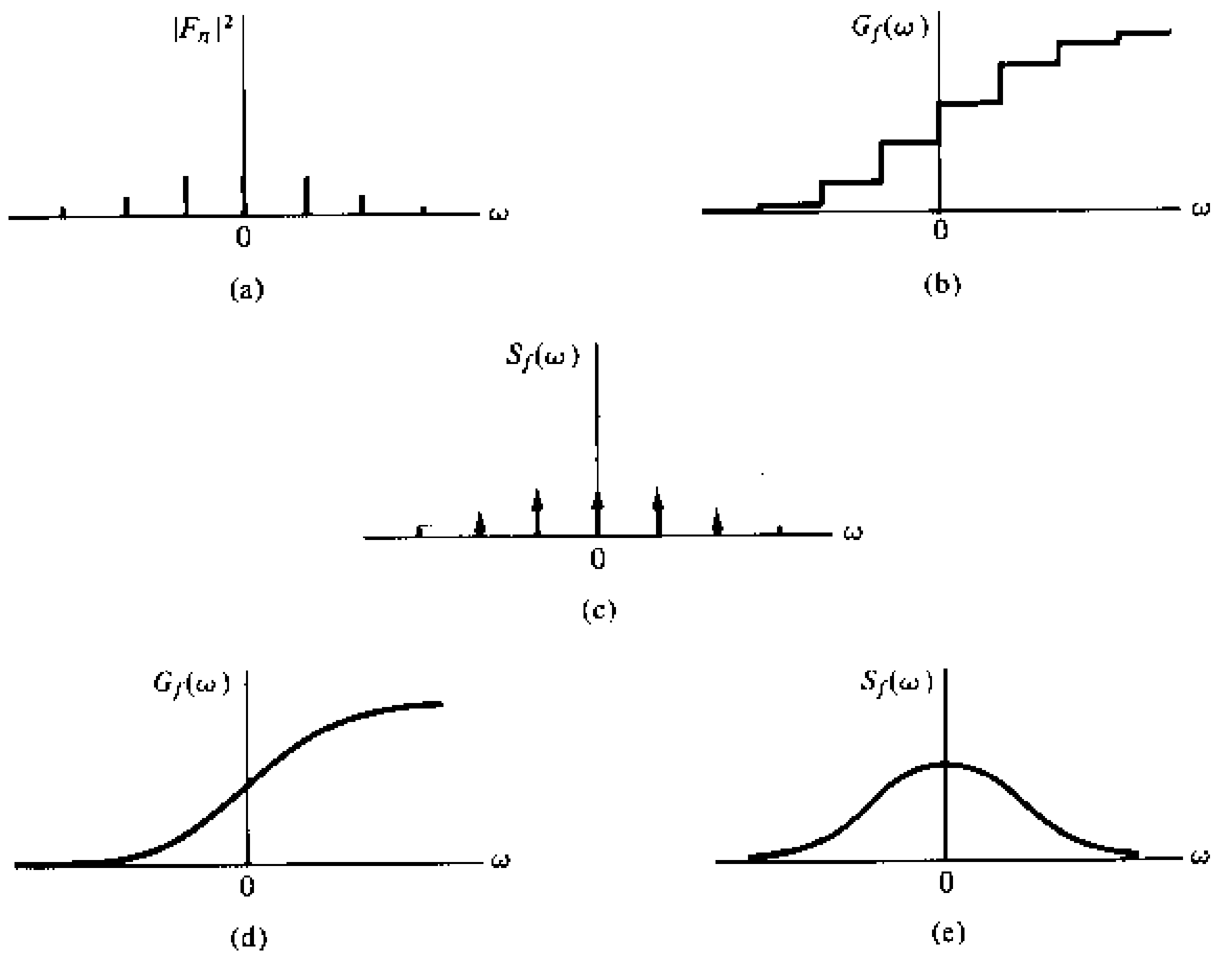


Figure 4.4 Power spectra of periodic functions: (a) line power spectrum of a periodic function; (b) integrated power spectrum of a periodic function; (c) power spectral density of a periodic function; (d) integrated power spectrum of an aperiodic function; and (e) power spectral density of an aperiodic function.

Example:

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$x(t) = \underbrace{\left(\frac{A}{2}e^{j\theta}\right)}_{X_{-1}} e^{j\omega_0 t} + \underbrace{\left(\frac{A}{2}e^{-j\theta}\right)}_{X_1} e^{-j\omega_0 t}$$

$X_1,\ X_{-1}:$ Fourier Series coefficients

$$X(\omega) = 2\pi X_{-1}\delta(\omega + \omega_0) + 2\pi X_1 \delta(\omega - \omega_0)$$

$$S_x(\omega) = 2\pi |X_{-1}|^2 \delta(\omega + \omega_0) + 2\pi |X_1|^2 \delta(\omega - \omega_0)$$

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega = \frac{A^2}{2}$$

The same result was obtained previously using time-domain tools.

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) \qquad Y(\omega) = H(\omega)X(\omega)$$

$|H(\omega)|$: magnitude of the transfer function

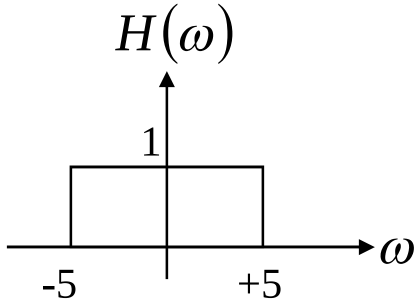
Energy spectral density of the output signal $\quad |Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$

Energy of the output signal $\quad E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 |X(\omega)|^2 d\omega$

Power spectral density of the output signal $\quad S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

Power of the output signal $\quad P_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega) d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega$

Example: A voltage signal is described by $x(t) = e^{-5t}u(t)$
It is applied to the input of an ideal low-pass filter. The gain of the filter is unity, the bandwidth is 5 rad/sec and the resistance levels are 50 Ω.
Calculate the energy of the input signal and of the output signal.



$$H(\omega) = \begin{cases} 1, & |\omega| < 5 \\ 0, & |\omega| > 5 \end{cases} \qquad X(\omega) = \frac{1}{\omega^2 + 25}$$

$$E_x = \frac{1}{R}\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{50}\int_{0}^{\infty} e^{-10t} dt = 2 \cdot 10^{-3} \ [\text{joule}]$$

$$E_y = \frac{1}{2\pi R}\int_{-\infty}^{\infty} |H(\omega)|^2 |X(\omega)|^2 d\omega = \frac{1}{2\pi 50}\int_{-5}^{+5} \frac{d\omega}{\omega^2 + 25} = 1 \cdot 10^{-3} \ [\text{joule}]$$

Autocorrelation Function

$$S_x(\omega) = \lim_{T\to\infty} \frac{\left|X_T(\omega)\right|^2}{T}$$

Taking Inverse Fourier Transform of PSD

$$\mathrm{F}^{-1}\{S_x(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty}\frac{1}{T}\left|X_T(\omega)\right|^2 e^{j\omega\tau}\,d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{+\infty} X_T^*(\omega)\,X_T(\omega)\,e^{j\omega\tau}\,d\omega$$

Replace by Fourier Transform expressions

$$= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{+\infty}\left[\int_{-T/2}^{T/2} x^*(t)e^{j\omega t}dt\right]\left[\int_{-T/2}^{T/2} x(u)e^{-j\omega u}du\right]e^{j\omega\tau}\,d\omega$$

Changing the order of integration

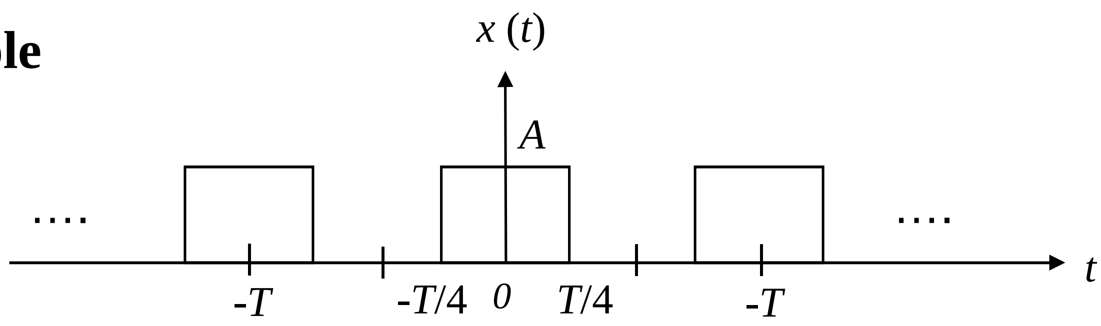
$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\int_{-T/2}^{T/2} x(u)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-u+\tau)}d\omega\right]du\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\int_{-T/2}^{T/2} x(u)\delta(t-u+\tau)\,du\,dt \quad = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^*(t)x(t+\tau)\,dt$$

$$\boxed{\mathrm{F}^{-1}\{S_x(\omega)\} \quad = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^*(t)x(t+\tau)\,dt \quad = R_x(\tau)}$$

Autocorrelation function of $x(t)$

Example



$$R_x(\tau)=\frac{1}{T}\int_{-T/2}^{T/2}x^*(t)x(t+\tau)dt$$

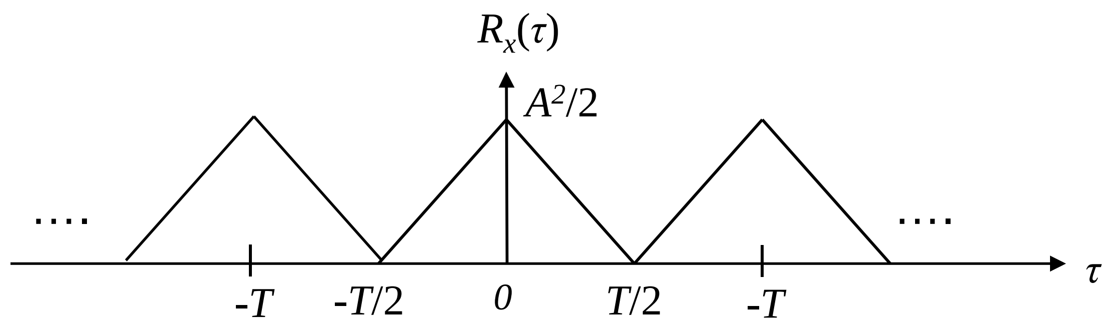
Since $x(t)$ is periodic, the limiting operation can be omitted.

$-T/2<\tau<0$

$$R_x(\tau)=\frac{1}{T}\int_{-T/4}^{T/4+\tau}A^2dt=A^2\left(\frac{1}{2}+\frac{\tau}{T}\right)$$

$0<\tau<T/2$

$$R_x(\tau)=\frac{1}{T}\int_{\tau-T/4}^{T/4}A^2dt=A^2\left(\frac{1}{2}-\frac{\tau}{T}\right)$$



Autocorrelation function is widely used in signal analysis. It is especially useful for the detection or recognition of signals that are masked by additive noise.

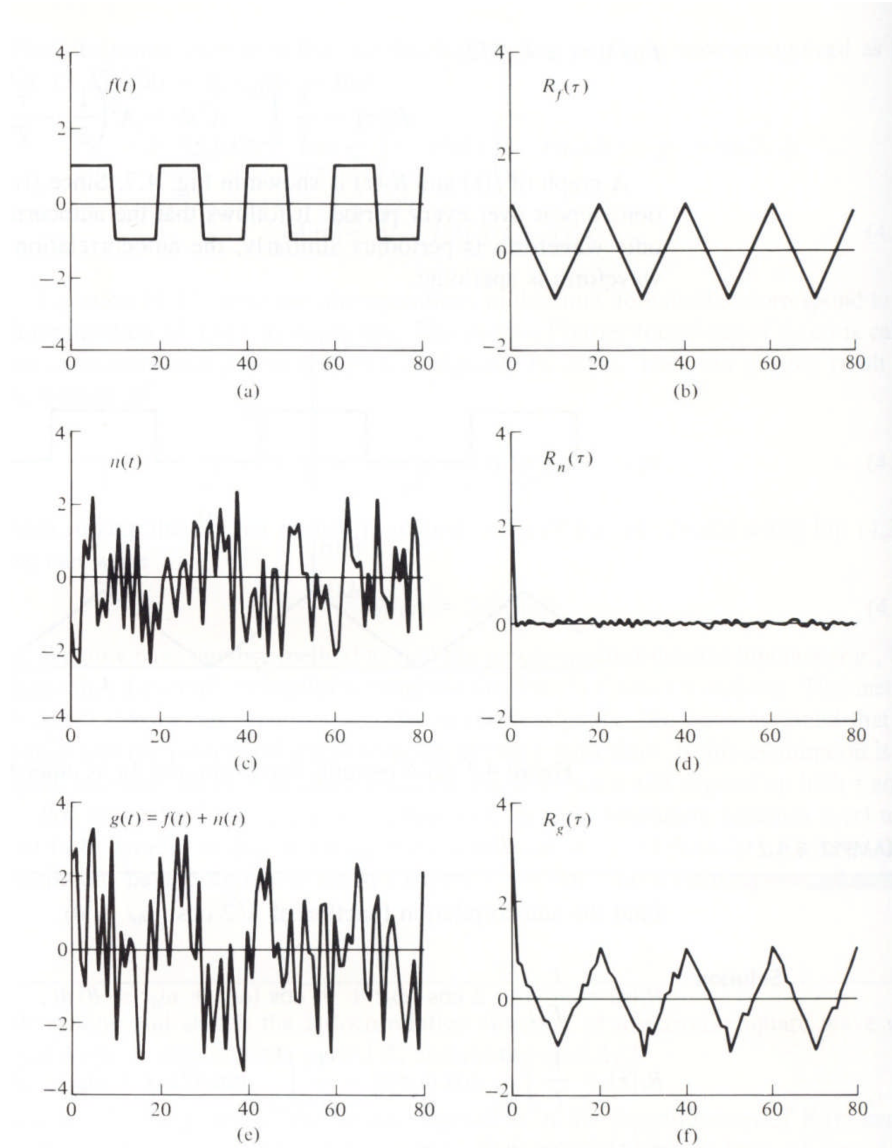
a) Original signal

c) Noise

e) Original signal + noise

b), d) and f) : corresponding autocorrelation functions

The autocorrelation function of the original signal is still recognizable in the noisy case!



$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(u)x(t-u)du$$

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)y^*(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\int_{-\infty}^{\infty} h(u)x(t-u)du \right]\left[\int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)dv \right]dt$$

Variable change $s=t-u$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)\left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(s)x^*(s+u-\tau-v)ds \right]dudv$$

Definition of R_x

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)R_x(\tau+v-u)dudv$$

Definition of convolution

$$= \int_{-\infty}^{\infty} [R_x(\tau+v) \otimes h(\tau+v)]h^*(v)dv$$

$$\boxed{R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h^*(-\tau)}$$

Time-domain

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = h(t) \otimes x(t)$$

$$R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h^*(-\tau)$$

Frequency-domain

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

Properties of Autocorrelation Functions

- **Symmetry**

$$R_x^*(\tau) = R_x(-\tau)$$

- **Mean-Square Value**

$$R_x(\tau)\Big|_{\tau=0} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = P_x$$

- **Periodicity**

If $x(t+T) = x(t)$, i.e. $x(t)$ periodic, $R_x(\tau+T) = R_x(\tau)$

- **Maximum Value**

The autocorrelation function is bounded by its mean square value.

$$|R_x(\tau)| \le R_x(0)$$

Cross-Correlation Function

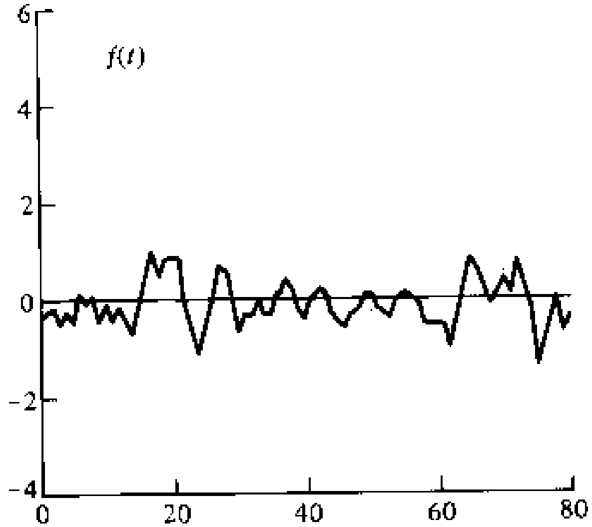
$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) y(t+\tau) dt$$

$R_{xy}(\tau) = 0 \longrightarrow$ $x(t)$ and $y(t)$ are uncorrelated.

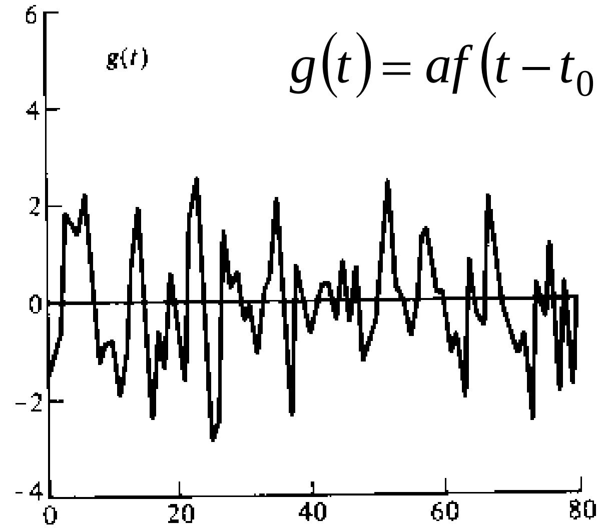
Example: $z(t) = x(t) + y(t)$ $\quad R_z(\tau) = ?$

$$R_z(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[x^*(t) + y^*(t)\right] \left[x(t) + y(t)\right] dt$$

$$= R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

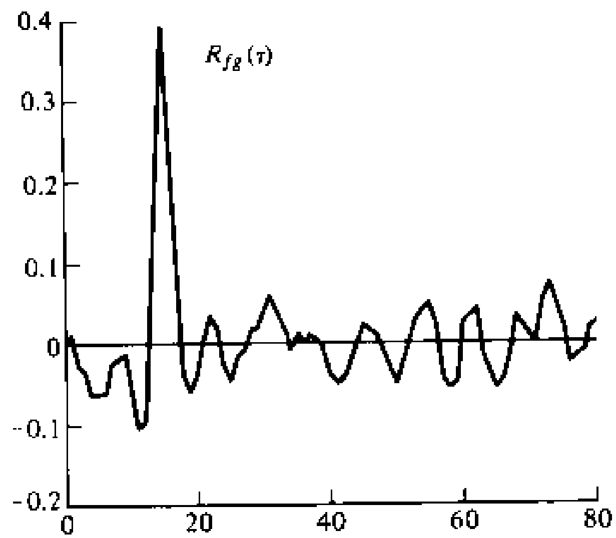


(a) Random signal



(b) Random signal + noise

$$g(t) = af(t - t_0) + n(t)$$



(c) Crosscorrelation

Cross-correlation can be used to find the time delay.

Typically used in synchronization for communication systems.

Noise

Noise consists of any unwanted signals that tend to disturb the transmission and processing of desired signals in communication system. Noise may be random or deterministic.

Noise sources:

external --- atmospheric noise, man made noise, etc.

internal --- thermal noise

Thermal noise is produced as a result of the thermally excited random motion of free electrons in a conducting medium, such as a resistor.

In the case of random noise, time-average representations are useful.

Mean (DC) value

$$\overline{n(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)\,dt$$

Bar indicates time averaging

Mean-square value (=Power)

$$P_n = \overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2\,dt$$

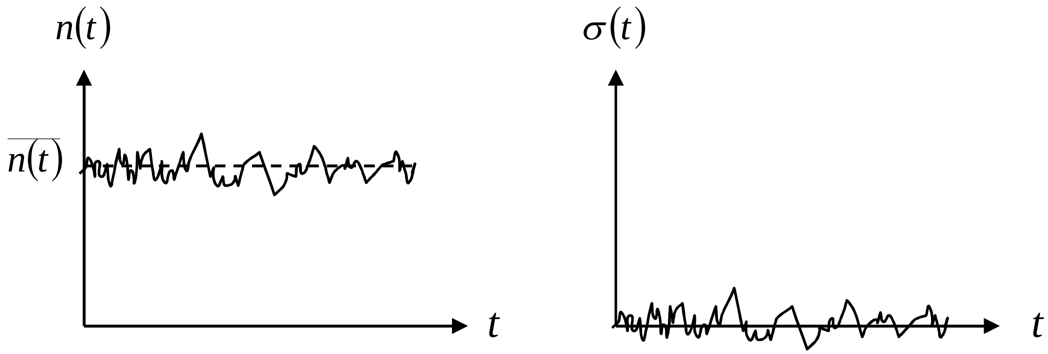
$\sqrt{\overline{n^2(t)}}$: root-mean-square (rms)

AC component

$$\sigma(t) \overset{def}{=} n(t) - \overline{n(t)}$$

Signal-to-noise ratio (SNR) is also widely used as a performance measure.

$$\frac{S}{N} = \frac{\overline{s^2(t)}}{\overline{\overline{n^2(t)}}} = \frac{\text{signal power}}{\text{noise power}} \qquad \left[\frac{S}{N}\right]_{dB} = 10\log_{10}\left(\frac{\overline{s^2(t)}}{\overline{\overline{n^2(t)}}}\right)$$



$\overline{n(t)}$: constant

$\overline{\sigma(t)} = 0$

$$\sigma(t) = n(t) - \overline{n(t)}$$

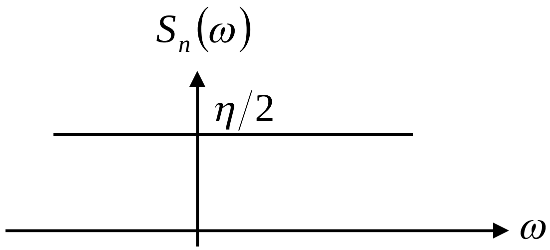
$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \overline{n(t)} + \sigma(t) \right|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \overline{n(t)} \right|^2 dt + \cancelto{0}{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \overline{n(t)}\sigma^*(t) dt} + \cancelto{0}{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\overline{n(t)}\right]^* \sigma(t) dt} + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sigma(t) \right|^2 dt$$

$$= \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \overline{n(t)} \right|^2 dt}_{\text{DC power}} + \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sigma(t) \right|^2 dt}_{\text{AC power}}$$

Band-Limited White Noise

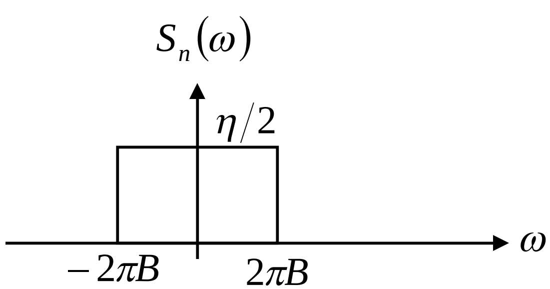
A flat power spectrum contains all frequency components with equal power weighting and is designated as *white*, in an analogy to white light.



$$P_n = \overline{n^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)\,d\omega \to \infty$$

Infinite power!

In general, the bandwidth of the receiver (i.e. front end filter) is narrower than the bandwidth limitations of the noise process. If noise has a flat PSD extending beyond the bandwidth of a given system, the noise appears to the system as if it were band-limited and white.



$$P_n = \overline{n^2(t)} = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \frac{\eta}{2}\,d\omega = \eta B$$

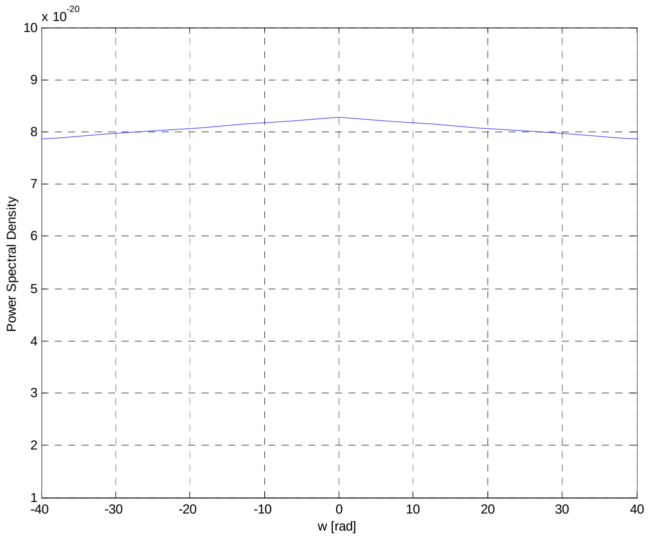
Thermal Noise

$$S_n(\omega) = \frac{h|\omega|}{\pi\left[\exp\left(\frac{h|\omega|}{2\pi kT}\right) - 1\right]}$$

T: temperature [K]

k: Boltzmann's constant (=1.38×10^{-23} joule/K)

h: Planck's constant (=6.625×10^{-34} joule-sec)



- Achieves its maximum at $\omega = 0$. The value of this maximum is $2kT$.
- $\omega \to \infty \quad PSD \to 0$

The rate of convergence is very slow.

$$S_n(\omega) \cong 2kT \qquad |\omega| << 2\pi kT/h$$

- $kT/h \approx 6 \cdot 10^{12}$ Hz Beyond the operating frequency of conventional communication systems!

Conclusion: Thermal noise can be assumed white for all practical purposes.