Angle Modulation

(Ch.6 in Textbook)

Objectives:

- To study frequency modulation (FM)
- To study phase modulation (PM)
- To study the structure of modulator and demodulator

Amplitude modulation (AM): The amplitude of the carrier varies in accordance with the message signal $f(t)$.

Angle modulation (FM, PM): The angle of the carrier is varied in accordance with the message signal while the amplitude of the carrier is constant.

$$\phi(t) = a(t)\cos(\omega_c t + \gamma(t))$$

AM FM, PM

Amplitude Modulation vs. Angle Modulation

Amplitude Modulation:

-- The spectrum of the modulated signal is shifted/scaled version of the message signal spectrum.

-- Transmission bandwidth ≥ Message bandwidth

-- SNR_{out} can be increased only increasing the transmitted signal power.

Angle Modulation:

-- The spectrum of the modulated signal is NOT simply related to message signal spectrum.

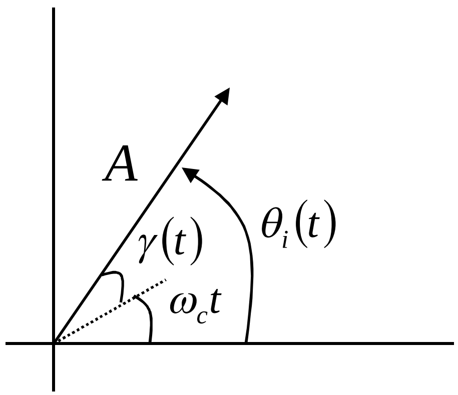
-- Transmission bandwidth >>Message bandwidth.

-- Improvement in SNR_{out} without increasing the transmitted signal power

Angle Modulated Signal

$$\phi(t) = A\cos(\theta_i(t)) = A\cos(\omega_c t + \gamma(t))$$

Phasor Representation: Signal is represented by a rotating vector in complex plane.



$$\phi(t) = A\operatorname{Re}\left\{e^{j\theta_i(t)}\right\} = A\operatorname{Re}\left\{e^{j[\omega_c t + \gamma(t)]}\right\}$$

A: phasor magnitude

$\theta_i(t)$: instantaneous phase angle (determines the position of rotating vector at time t)

$$\theta_i(t) = \omega_c t + \underbrace{\gamma(t)}_{\text{related to message signal}}$$

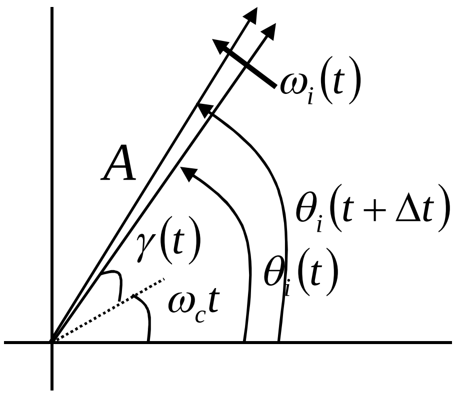
Angular velocity:
(frequency)

$$\phi(t) = A\cos(\theta_i(t))$$

$$t_1 = t \longrightarrow \theta_i(t)$$

$$t_2 = t + \Delta t \longrightarrow \theta_i(t+\Delta t)$$

$$\omega_{\Delta t}(t) \overset{def}{=} \frac{\theta_i(t+\Delta t) - \theta_i(t)}{\Delta t}$$



<u>average</u> frequency for duration of Δt
indicates how fast/slow angle changes

$$\omega_i(t) \overset{def}{=} \lim_{\Delta t \to 0} \omega_{\Delta t}(t) = \lim_{\Delta t \to 0} \frac{\theta_i(t+\Delta t) - \theta_i(t)}{\Delta t} = \frac{d\theta_i(t)}{dt}$$

<u>instantenous</u> frequency at time *t*
also known as "*angular velocity*"

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + \frac{d\gamma(t)}{dt}$$

varied with message signal

Recall

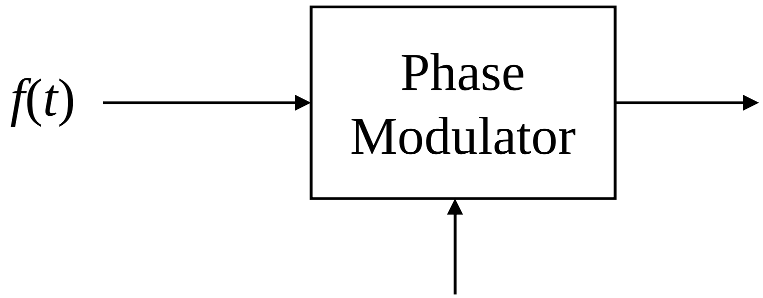
X=vt

X=distance, v=velocity, t= time

Angle: $\theta_i(t) = \int_0^t \omega_i(\tau)d\tau + \theta_0$

Phase Modulation (PM)

$$\phi(t) = A_c \cos(\omega_c t + \gamma(t))$$ → generic form



$$\phi_{PM}(t) = A_c \cos(\omega_c t + k_p f(t) + \theta_0)$$

k_p: phase deviation constant

$$c(t) = A_c \cos(\omega_c t + \theta_0)$$

Instantenous angle $\theta_i(t) = \omega_c t + k_p f(t) + \theta_0$ → Phase $\propto f(t)$ → **PM**

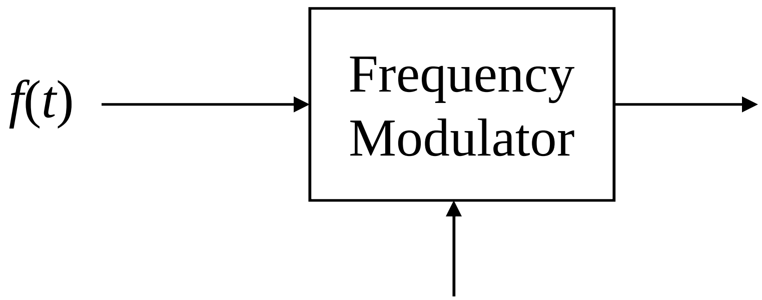
Instantenous frequency $\omega_i(t) = \dfrac{d\theta_i(t)}{dt} = \omega_c + k_p \dfrac{df(t)}{dt}$

$$\Delta\phi_{\max} = k_p \max(|f(t)|)$$

→ Maximum (peak) phase deviation

Frequency Modulation (FM)

$$\phi(t) = A_c \cos(\omega_c t + \gamma(t))$$ → generic form



$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_0 \right)$$

k_f: frequency deviation constant

$$c(t) = A_c \cos(\omega_c t + \theta_0)$$

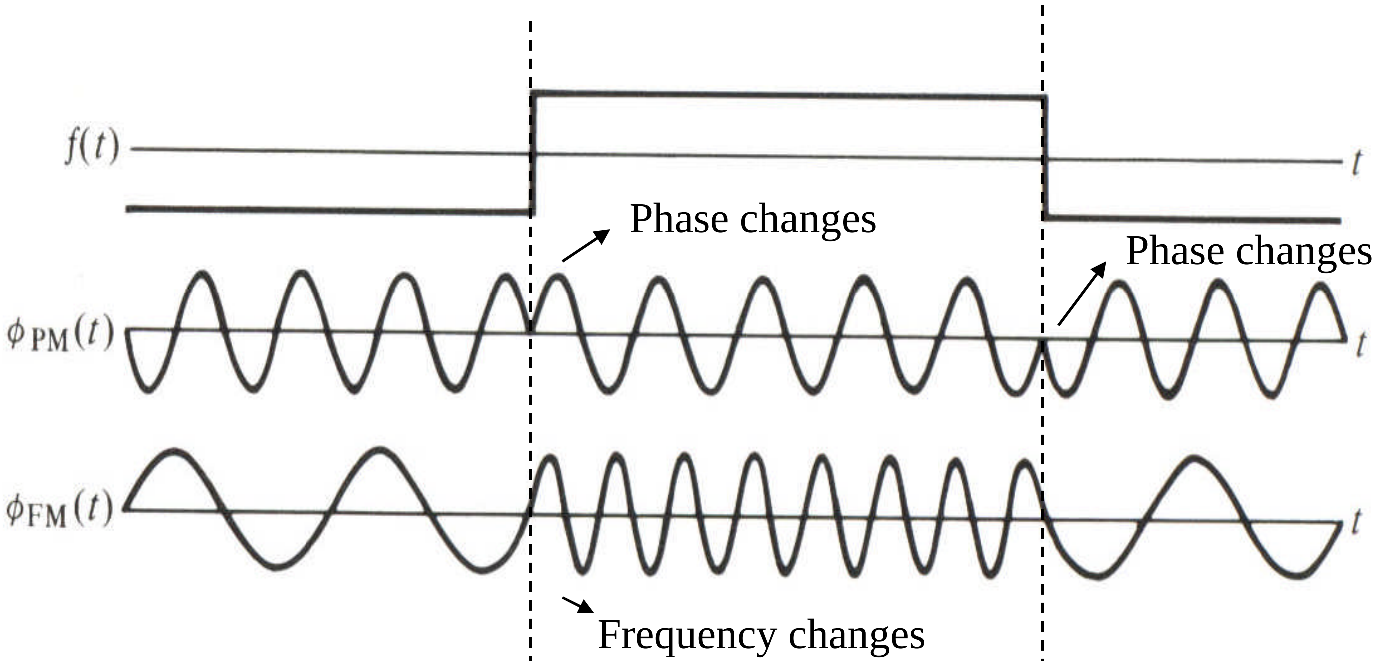
Instantenous angle $$\theta_i(t) = \omega_c t + k_f \int_0^t f(\tau) d\tau + \theta_0$$

Instantenous frequency $$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_f f(t)$$ → Frequency $\propto f(t)$ → **FM**

$$\Delta f_{\max} = k_f \max(|f(t)|)$$

→ Maximum (peak) frequency deviation

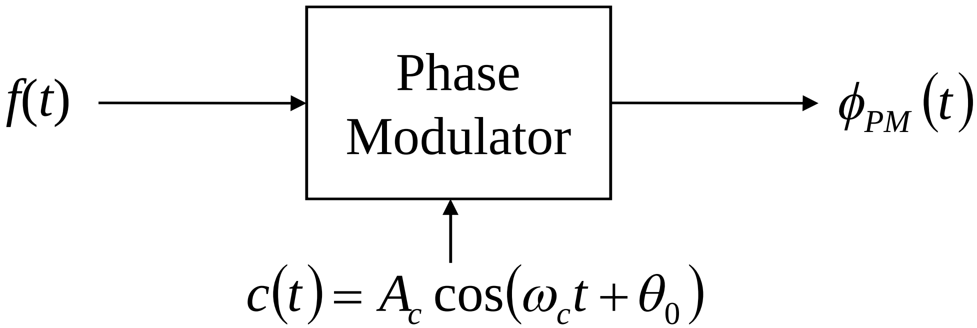
Example of FM and PM signals

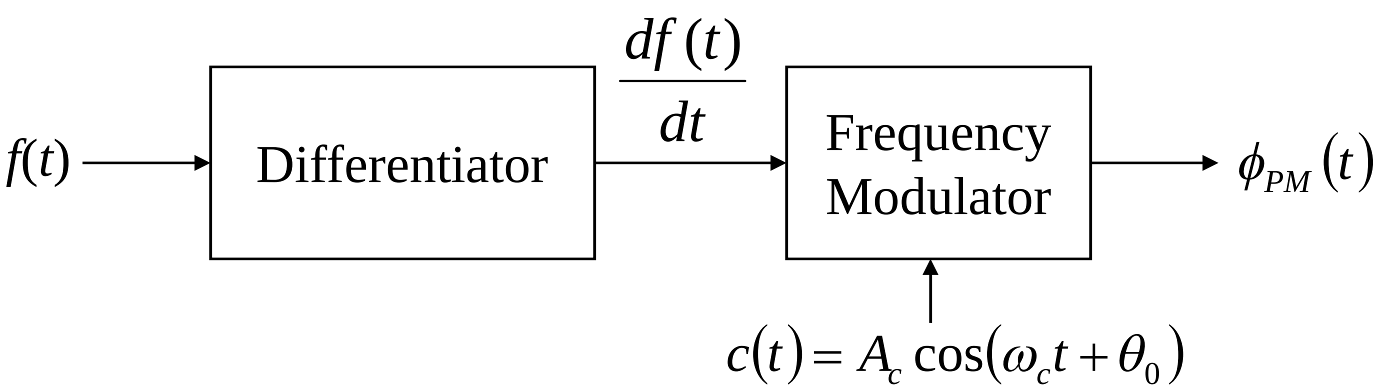


Relation between FM and PM Signals (I)

$$\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p f(t) + \theta_0\right)$$

$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau)\,d\tau + \theta_0\right)$$

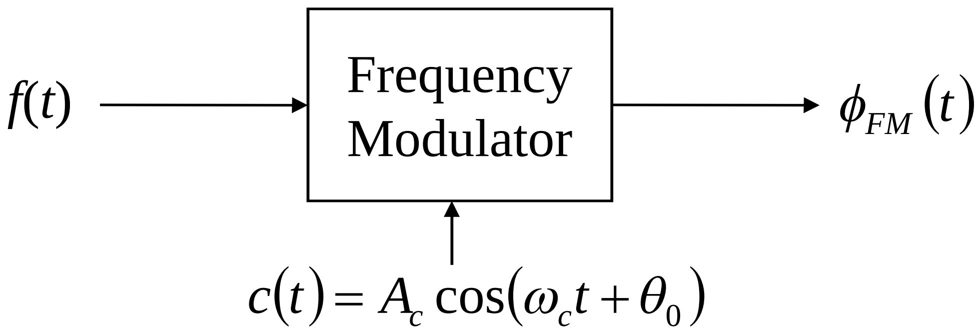


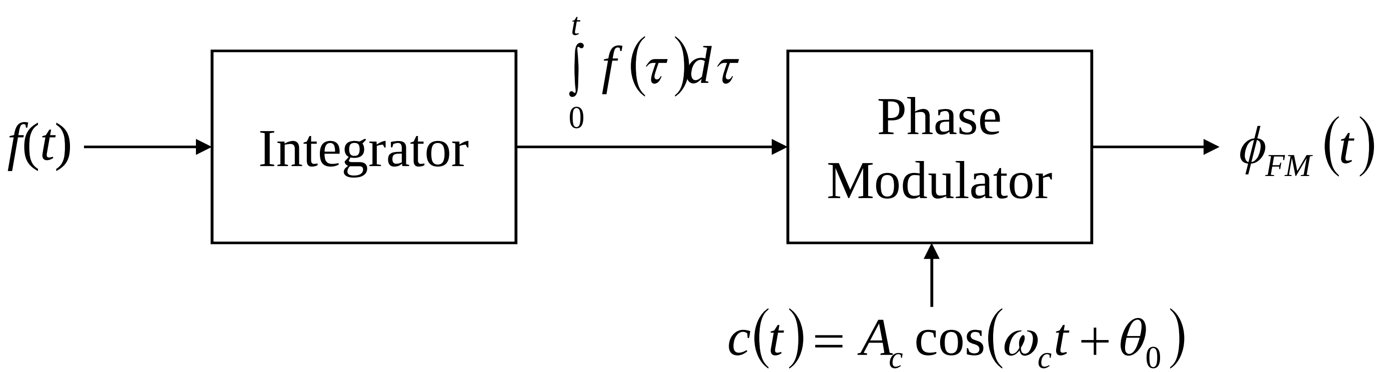


Relation between FM and PM Signals (II)

$$\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p f(t) + \theta_0\right)$$

$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau)d\tau + \theta_0\right)$$





Narrowband FM (NBFM)

$$\phi(t) = A_c \cos(\omega_c t + \gamma(t))$$
$$= A_c \cos(\omega_c t)\cos(\gamma(t)) - A_c \sin(\omega_c t)\sin(\gamma(t))$$

$\cos(a+b)$
$= \cos a \cos b - \sin a \sin b$

NBFM assumption $|\gamma(t)| << 1$

$\cos(\gamma(t)) \approx 1 \quad \sin(\gamma(t)) \approx \gamma(t)$

$$\phi(t) \approx A_c \cos(\omega_c t) - A_c \gamma(t) \sin(\omega_c t)$$

→ NBFM signal

<u>A Case Study:</u> NBFM Single Tone Modulation

Assume $f(t) = a\cos(\omega_m t)$ Single Tone Modulation

$$\phi(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau)d\tau + \theta_0\right) \qquad \theta_0 = 0$$

Instantenous frequency

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_f f(t) = \omega_c + ak_f \cos(\omega_m t)$$
$$= \omega_c + \Delta\omega \cos(\omega_m t)$$

$$\Delta\omega = k_f \max(|f(t)|) = ak_f$$ Peak frequency deviation

Instantenous angle

$$\theta_i(t) = \omega_c t + k_f \int_0^t f(\tau)d\tau = \omega_c t + ak_f \int_0^t \cos(\omega_m \tau)d\tau$$
$$= \omega_c t + \frac{ak_f}{\omega_m}\sin(\omega_m t)$$

$$\beta \overset{def}{=} \frac{\Delta\omega}{\omega_m} = \frac{a\ k_f}{\omega_m}$$ Modulation index of FM signal

A Case Study: NBFM Single Tone Modulation (Cont'd)

$$f(t) = a\cos(\omega_m t)$$

$$\phi(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau) d\tau\right)$$

$$= A_c \cos(\omega_c t + \beta \sin(\omega_m t))$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$= A_c \cos(\omega_c t)\cos[\beta \sin(\omega_m t)] - A_c \sin(\omega_c t)\sin[\beta \sin(\omega_m t)]$$

NBFM $\quad |\beta \sin(\omega_m t)| << 1 \longrightarrow |\beta| << 1$

$$\cos[\beta \sin(\omega_m t)] \approx 1 \qquad \sin[\beta \sin(\omega_m t)] \approx \beta \sin(\omega_m t)$$

$$\phi(t) \approx A_c \cos(\omega_c t) - A_c \beta \sin(\omega_m t)\sin(\omega_c t)$$

→ NBFM signal for single tone modulation

<u>A Case Study:</u> NBFM Single Tone Modulation (Cont'd)

$$\phi(t) \approx A_c \cos(\omega_c t) - A_c \beta \sin(\omega_m t) \sin(\omega_c t)$$

Rearranging as $\phi(t) = k \cos(\omega_c t + \alpha)$

$$k = A_c \sqrt{1 + \beta^2 \sin(\omega_m t)}$$

$$\alpha = \tan^{-1}(\beta \sin(\omega_m t))$$

$$\phi(t) = \underbrace{A_c \sqrt{1 + \beta^2 \sin^2(\omega_m t)}}\cos\left\{\omega_c t + \tan^{-1}[\beta \sin(\omega_m t)]\right\}$$

ω_m modulates the amplitude

$\longrightarrow$ the envelope suffers distortion, unless β is very small

<u>A Case Study:</u> NBFM Single Tone Modulation (Cont'd)

<u>The limits of β for NBFM approximation</u>

$$\phi(t) \approx A_c \sqrt{1+\beta^2 \sin^2(\omega_m t)} \cos\{\omega_c t + \tan^{-1}[\beta \sin(\omega_m t)]\}$$

$$\alpha = \tan^{-1}(\beta \sin(\omega_m t))$$

$$\frac{d\alpha}{dt} = \frac{\beta\omega_m \cos(\omega_m t)}{1+\beta^2 \sin^2(\omega_m t)} \stackrel{?}{=} \Delta\omega \cos(\omega_m t)$$

$$\approx \beta\omega_m \cos(\omega_m t) \quad \text{if } \beta^2 \sin^2(\omega_m t) << 1$$

$$A_c \sqrt{1+\beta^2 \sin^2(\omega_m t)} \approx A_c \qquad \text{if } \beta^2 \sin^2(\omega_m t) << 1$$

$$\left.\begin{array}{l} \beta^2 \sin^2(\omega_m t) << 1 \\ \sin^2(\omega_m t) \leq 1 \end{array}\right\} \beta^2 < 1 \longrightarrow \beta < 0.316$$

A reasonable bound for the narrowband approximation

Wideband FM (WBFM)

For a general FM signal, it is not possible to evaluate the Fourier transform. Here, we again focus on single tone modulation.

$$f(t) = a\cos(\omega_m t)$$

$$\phi(t) = A_c \cos\left(\omega_c t + k_f \int_0^t f(\tau)d\tau\right) = A_c \cos\left(\omega_c t + ak_f \int_0^t \cos(\omega_m \tau)d\tau\right)$$

$$= A_c \cos(\omega_c t + \beta \sin(\omega_m t)) \qquad \beta = \frac{ak_f}{\omega_m}$$

How can we write the above expression in terms of $\cos[(\omega_c + n\omega_m)t]$?

$$n = 0, \mp 1, \mp 2, \ldots.$$

WBFM (Cont'd)

$$\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$$

$$= A_c \operatorname{Re}\left\{e^{j[\omega_c t + \beta \sin(\omega_m t)]}\right\} = A_c \operatorname{Re}\left\{e^{j\omega_c t} x(t)\right\}$$

$$x(t) = e^{j\beta \sin(\omega_m t)} \qquad x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_m t} \longleftrightarrow X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_m t} dt$$

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} e^{j\beta \sin(\omega_m t)} e^{-jn\omega_m t} dt$$

Variable change

$\xi = \omega_m t = (2\pi/T)t$

$$X_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin \xi - n\xi)} d\xi$$

This integral can be evaluated numerically in terms of the parameters n and β and tabulated extensively. It is denoted by $J_n(\beta)$ and called *Bessel function of the first kind*. They are real-valued.

WBFM (Cont'd)

$$x(t) = e^{j\beta\sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{jn\omega_m t} \quad \text{where} \quad J_n(\beta) \overset{def}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin\xi - n\xi)} d\xi$$

$J_n(\beta) \rightarrow X_n$

$$\phi(t) = A_c \operatorname{Re}\left\{e^{j[\omega_c t + \beta\sin(\omega_m t)]}\right\} = A_c \operatorname{Re}\left\{e^{j\omega_c t} x(t)\right\}$$

$$= A_c \operatorname{Re}\left\{e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t}\right\} = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\phi(t) = A_c \cos(\omega_c t + \beta\sin(\omega_m t)) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\Phi(\omega) = \mathcal{F}[\phi(t)]$$

$$\Phi(\omega) = A_c \pi \sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Basic Properties of Bessel Functions

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

$$J_n(\beta) = \begin{cases} J_{-n}(\beta), & \text{if } n \text{ is even} \\ -J_{-n}(\beta), & \text{if } n \text{ is odd} \end{cases}$$

For $\beta << 1$

$$J_0(\beta) \approx 1$$

$$J_1(\beta) \approx \beta/2$$

$$J_n(\beta) \approx 0 \quad \text{for } n \geq 2$$

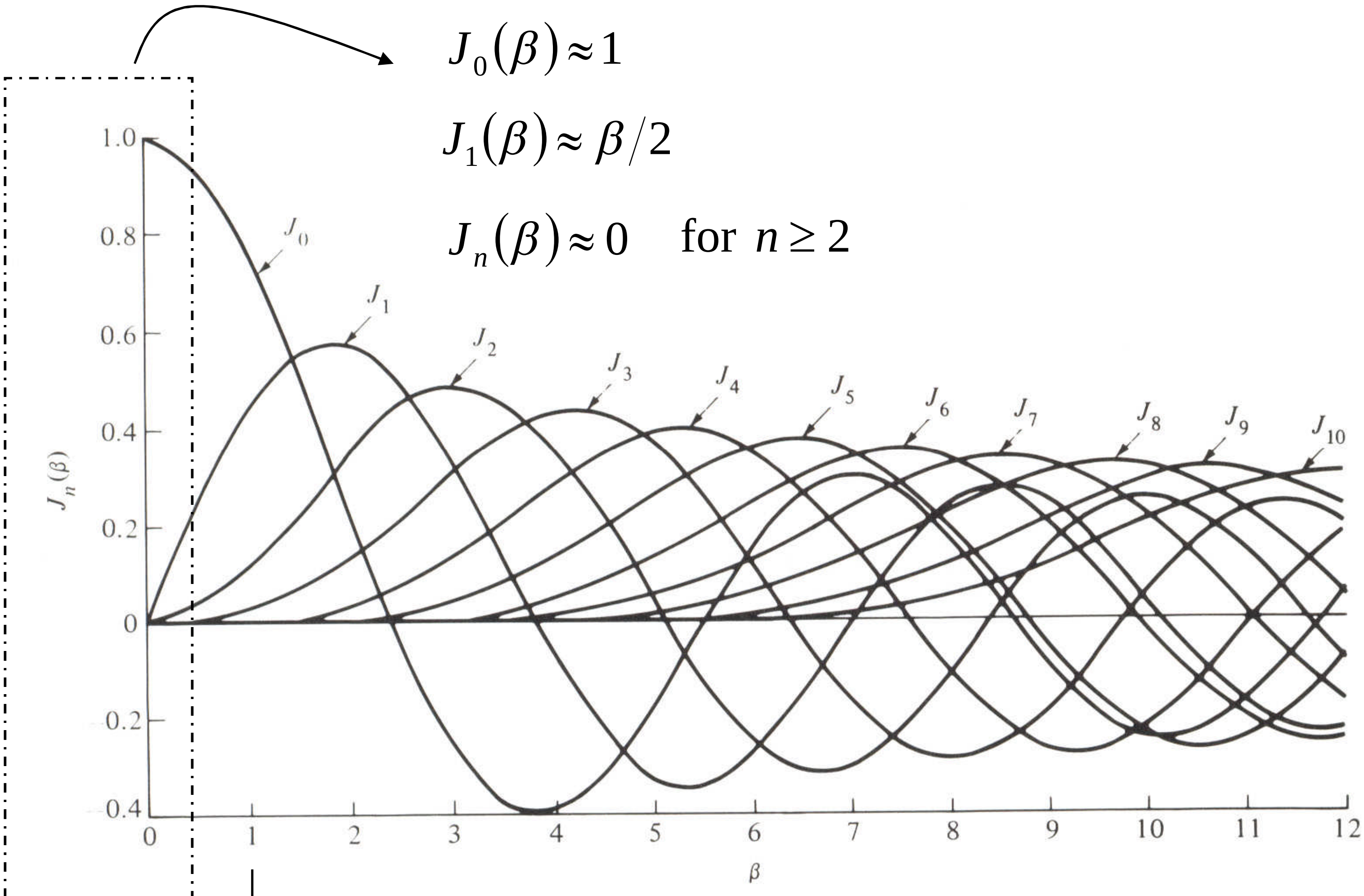
$$\sum_{n=-\infty}^{+\infty} J_n^2(\beta) = 1$$

Plots of Bessel function of the first kind

$$J_0(\beta) \approx 1$$

$$J_1(\beta) \approx \beta/2$$

$$J_n(\beta) \approx 0 \quad \text{for } n \geq 2$$



$\beta << 1$

$\beta = 1$

WBFM Signal

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

How can we obtain NBFM signal from WBFM expression?

For $\beta << 1$

$J_n(\beta) \approx 0$ for $n \geq 2$

$$\phi(t) \approx A_c J_0(\beta)\cos(\omega_c t) + A_c J_1(\beta)\cos[(\omega_c + \omega_m)t] + A_c J_{-1}(\beta)\cos[(\omega_c - \omega_m)t]$$

$J_0(\beta) \approx 1$ $\quad J_1(\beta) \approx \beta/2$ $\quad J_{-1}(\beta) \approx -\beta/2$

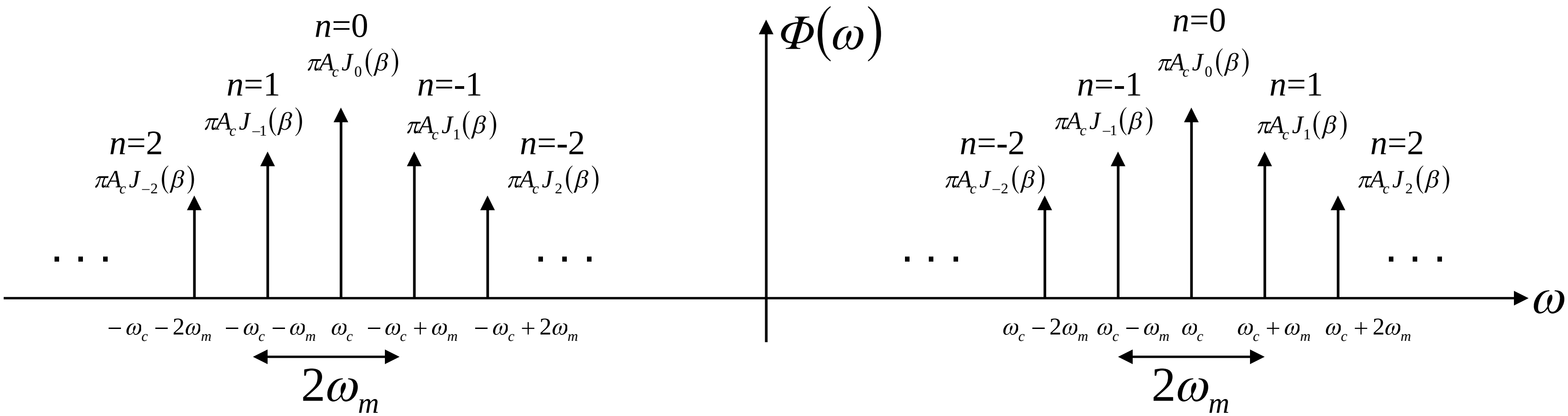
$$\longrightarrow \phi(t) \approx A_c \cos(\omega_c t) - A_c\beta \sin(\omega_m t)\sin(\omega_c t)$$

Bandwidth of FM Signals

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\longleftrightarrow \Phi(\omega) = A_c\pi \sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(\omega-\omega_c-n\omega_m)+\delta(\omega+\omega_c+n\omega_m)]$$

$\Phi(\omega)$ contains a carrier component and an infinite set of side frequencies located symmetrically on either side of carrier frequency (i.e. $\pm n\omega_m$). These frequencies are integer multiples of ω_m, defined as *harmonics*.



In general, $B(\text{rad}) = 2n\omega_m$ *n* depends on definition of *significant sidebands*

For $\beta << 1$, only *n*=0,±1 terms exist. $B(\text{rad}) = 2\omega_m$

"Significant Sidebands" Definition for Bandwidth of FM Signals

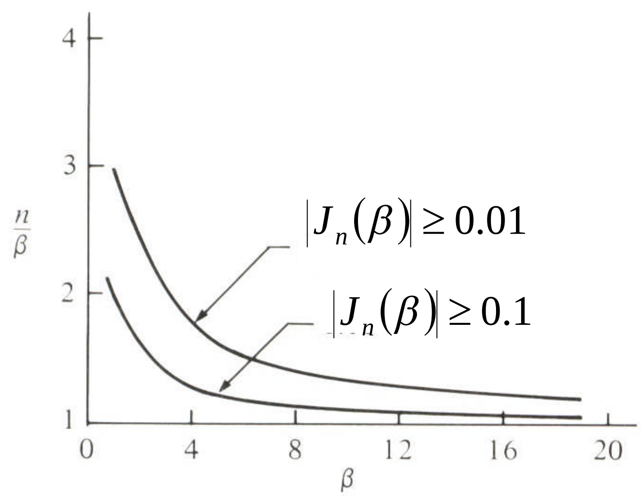
How many sidebands are important to the FM transmission of a signal?

- This depends on the intended application requirements and message signal.
- A rule commonly adopted is that a sideband is significant if its magnitude is equal to or exceeds 1% of the unmodulated carrier, i.e.

$$|J_n(\beta)| \geq 0.01$$

- $J_n(\beta)$ diminishes rapidly and the ratio $n/\beta \rightarrow 1$ as β becomes large.

$$\beta >> 1 \quad n \approx \beta$$



Therefore, the bandwidth for large β can be approximated as

$$B(\text{rad}) = 2n\omega_m \approx 2\beta\omega_m = 2\Delta\omega$$

Recall $\beta = \dfrac{\Delta\omega}{\omega_m} = \dfrac{ak_f}{\omega_m}$

Carson's Rule for Bandwidth of FM Signals

$$B(\text{rad}) \approx 2\omega_m(1+\beta)$$

$$B(\text{rad}) \approx 2(\omega_m + \Delta\omega)$$

can be neglected

For very small β, $B(\text{rad}) \approx 2\omega_m(1+\beta) \approx 2\omega_m$

can be neglected

For very large β, $B(\text{rad}) \approx 2\omega_m(1+\beta) \approx 2\omega_m\beta = 2\Delta\omega$

- Carson's Rule agrees with our previous observations for limiting cases, obtained for the special case of modulating signal in the form of a sinusoidal.
- Carson's Rule also holds for <u>general</u> modulating signals that are band-limited and have finite power.
- Carson's Rule gives less bandwidth than our definition of "significant sidebands". (not true for the limiting cases)

Average Power of FM Signals

$$\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$$

Total average power $P = \overline{\phi^2(t)} = A_c^2 / 2$

Alternative representation (in terms of Bessel function)

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) \cos[(\omega_c + n\omega_m)t]$$

$$\longleftrightarrow \; \Phi(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} \frac{A_c J_n(\beta)}{2} [\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Power spectral density

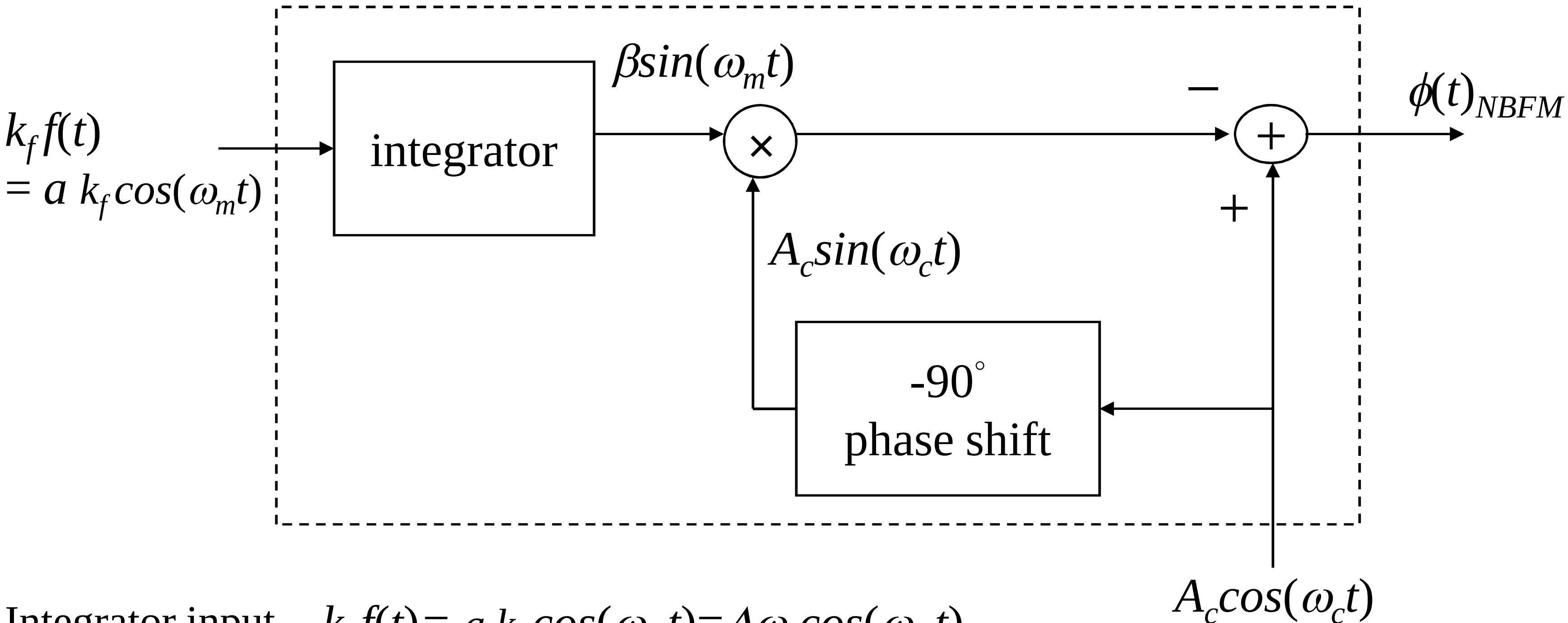
$$S_\phi(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} \left| \frac{A_c J_n(\beta)}{2} \right|^2 [\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Total average power $P = \overline{\phi^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\phi(\omega) d\omega = 2 \frac{A_c^2}{4} \underbrace{\sum_{n=-\infty}^{+\infty} J_n^2(\beta)}_{=1} = \frac{A_c^2}{2}$

Generation of NBFM Signals

FM Signal $\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$

Under NB assumptions $\phi(t) \approx A_c \cos(\omega_c t) - A_c \beta \sin(\omega_m t) \sin(\omega_c t)$

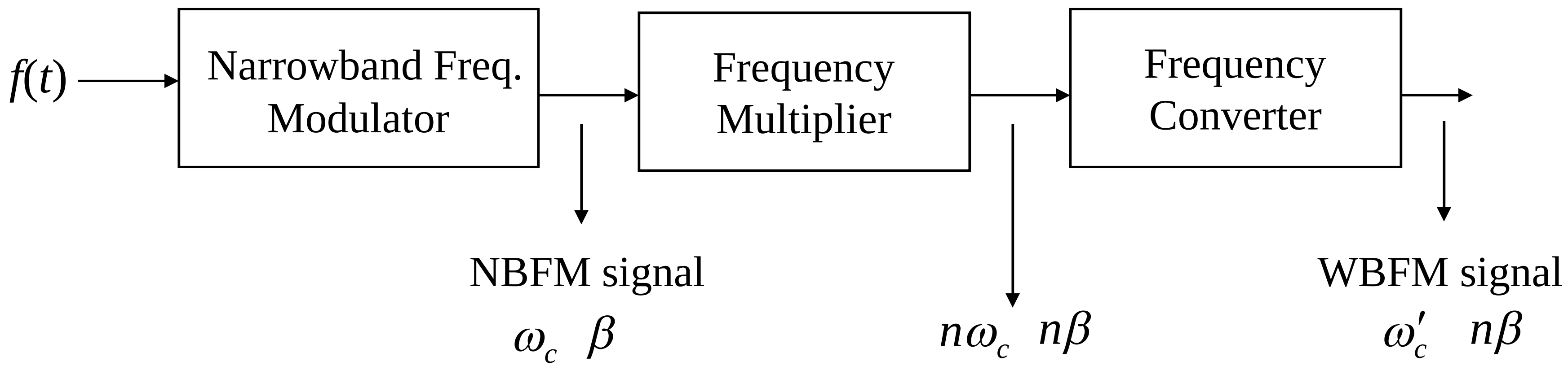


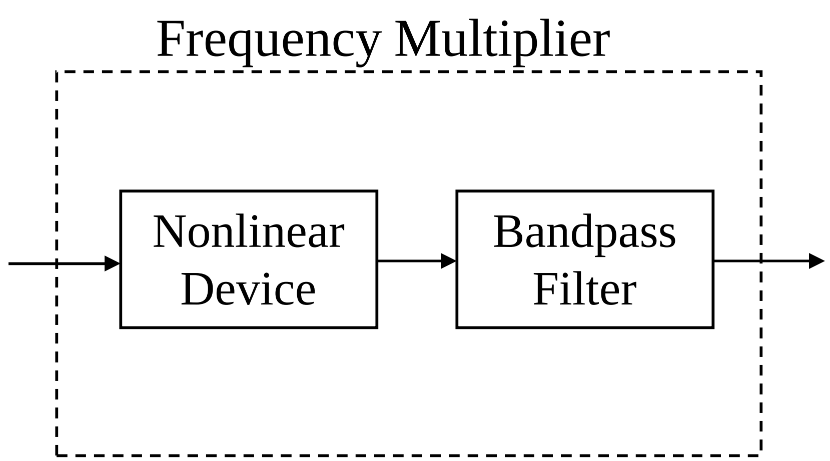
Integrator input $k_f f(t) = a\, k_f cos(\omega_m t) = \Delta\omega\, cos(\omega_m t)$

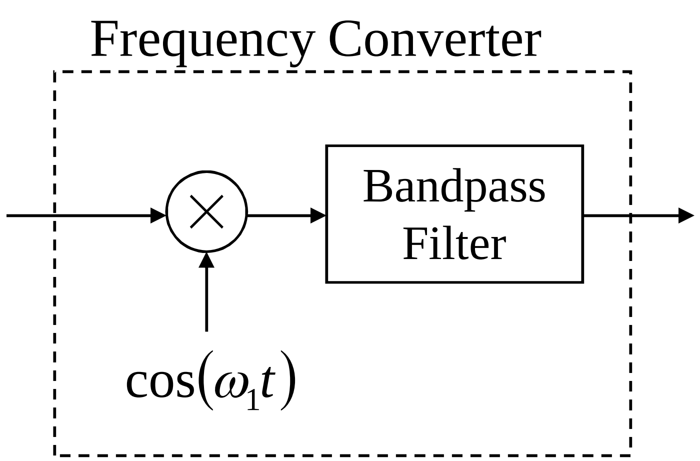
Integrator output $(\Delta\omega/\omega_m)\, sin(\omega_m t) = \beta\, sin(\omega_m t)$

Generation of WBFM Signals

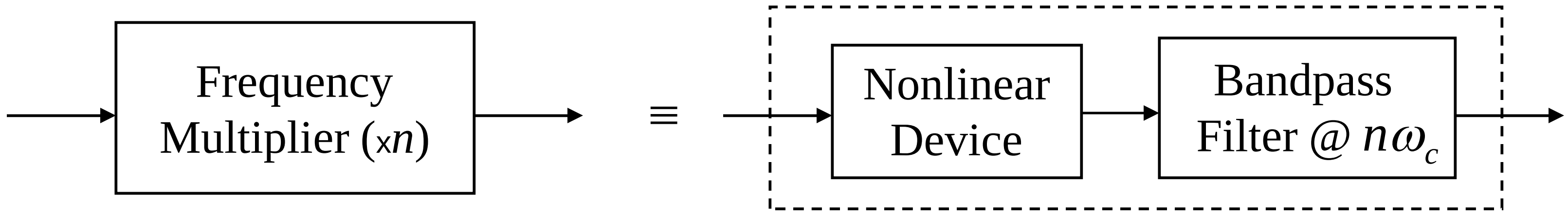
Indirect Method







Indirect Method (Cont'd)



Nonlinear device designed to multiply the frequencies of the input signal by a given factor.

$$e_{out}(t) = \sum_{k=0}^{n} c_k e_{in}^k(t)$$

Assume NBFM has mod. index β and desired WBFM has mod. index 2β

We need to use n=2 $\quad e_{out}(t) = c_0 + c_1 e_{in}(t) + c_2 e_{in}^2(t)$

Input $\quad e_{in}(t) = A\cos(\omega_c t + \beta \sin \omega_m t)$

Output $e_{out}(t) = c_0 + c_1 A\cos(\omega_c t + \beta \sin \omega_m t) + c_2 A^2 \cos^2(\omega_c t + \beta \sin \omega_m t)$

$$= \underbrace{c_0 + \frac{c_2 A^2}{2} + c_1 A\cos(\omega_c t + \beta \sin \omega_m t)}_{\text{Removed by the filter}} + \underbrace{\frac{c_2 A^2}{2}}_{\text{Can be adjusted by the filter gain}} \cos(\underline{2\omega_c}\, t + \boxed{2\beta} \sin \omega_m t)$$

Removed by the filter

Can be adjusted by the filter gain

Multiplied by 2

Indirect Method (Cont'd)

Considering the requirements on *n*, frequency multiplier might need to be implemented using multiple steps.

The output of the frequency multiplier:

$\cos\left(n\omega_c t + n\beta \sin \omega_m t\right)$ + additional terms

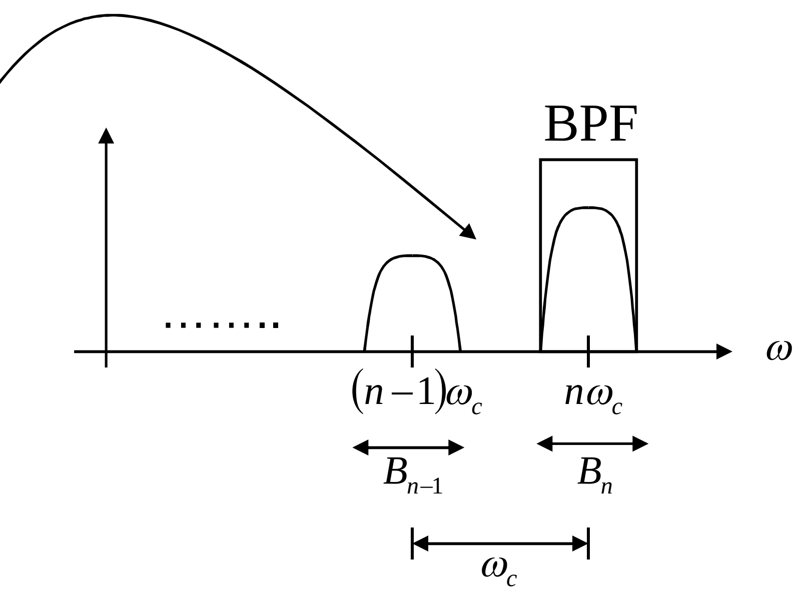
$$B_{n-1} = 2\left(\omega_m + (n-1)\Delta\omega\right)$$

$$B_n = 2\left(\omega_m + n\Delta\omega\right)$$

Limiting condition (no overlapping)

$$\frac{B_{n-1}}{2} + \frac{B_n}{2} < \omega_c$$

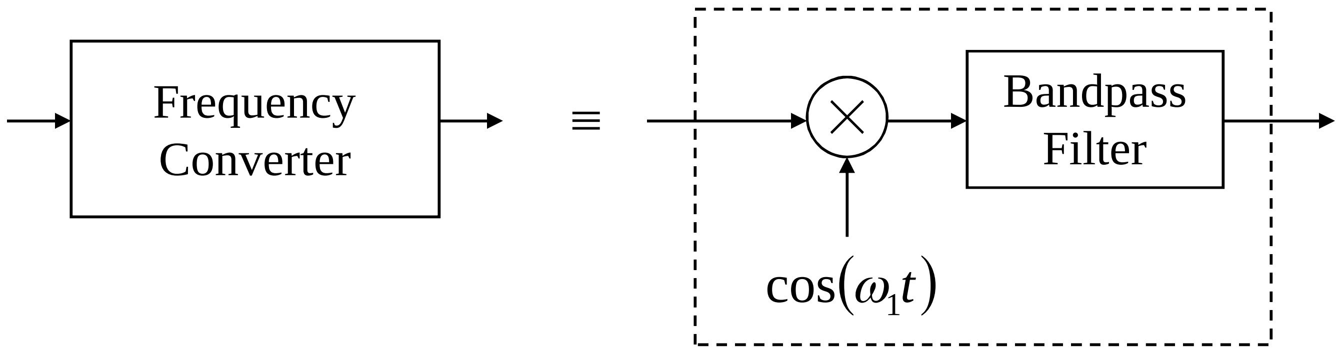
$$\Rightarrow \quad n < \frac{\omega_c - 2\omega_m}{2\Delta\omega} + \frac{1}{2}$$



Indirect Method (Cont'd)

Frequency multiplier output: $\cos\left(n\omega_c t + n\beta \sin \omega_m t\right)$

Frequency multiplier increases the modulation index by a factor of *n* as desired. This also results in an increase in the carrier frequency, which might not be sufficient in many cases. The carrier frequency needs to be shifted to the desired frequency by frequency converter.



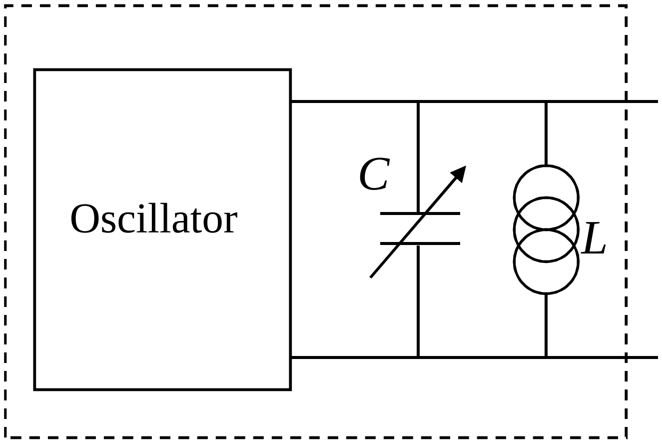
Frequency converter is used to shift the spectrum of the signal by a given amount. It does not change its spectral content.

One of the output terms should give desired frequency and the other will be removed by the proper choice of BPF. $n\omega_c - \omega_1$ $\quad n\omega_c + \omega_1$

Generation of WBFM Signals

Direct Method

Voltage-controlled oscillator (VCO) is any oscillator whose frequency is controlled by the modulating-signal voltage.



Oscillation frequency $\omega = \dfrac{1}{\sqrt{LC}}$

Voltage-variable capacitance $C = C_0 - kf(t)$

$$\omega = \frac{1}{\sqrt{LC_0}} \frac{1}{\sqrt{1 - kf(t)/C_0}} \approx \frac{1}{\sqrt{LC_0}} \left(1 + \frac{k}{2C_0} f(t)\right)$$

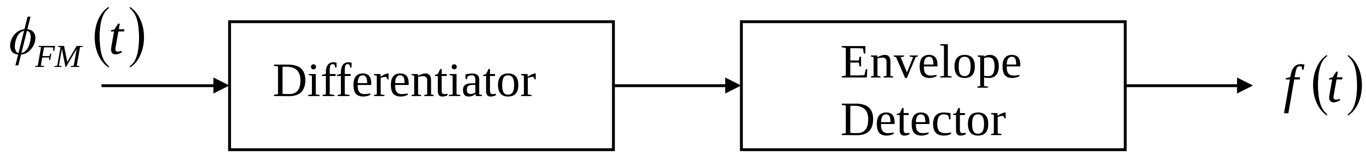
Recall:
$(1-x)^{-1/2} \approx 1 + x/2$

The frequency is proportional to the message signal $f(t)$.

The long-term frequency-stability is not as good as in the indirect method.

Demodulation of FM Signals

Direct Method: Discriminator



$$\phi_{FM}(t) = A\cos\left(\omega_c t + k_f \int_0^t f(\tau)d\tau\right)$$

$$\frac{d\phi_{FM}(t)}{dt} = -A\left[\omega_c + k_f f(t)\right]\sin\left(\omega_c t + k_f \int_0^t f(\tau)d\tau\right)$$

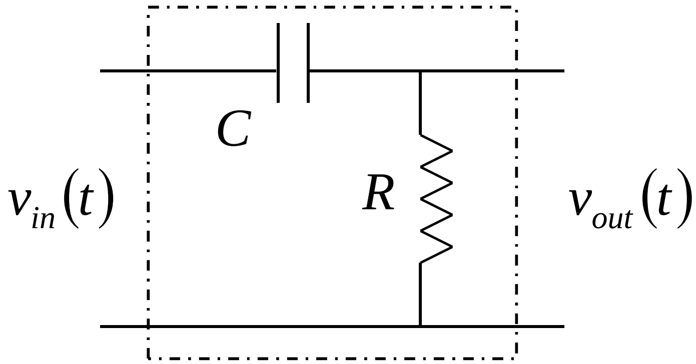
$k_f f(t) << \omega_c$ ⟶ The above expression has the form of DSB-LC signal.

Envelope of the signal at the output of differentiator $\quad A\omega_c\left[1 + \frac{k_f}{\omega_c} f(t)\right]$

There is a slight variation in the frequency. However, the envelope detector can be still used to detect the *f*(*t*).

The ideal differentiator can be approximated by any device whose magnitude transfer function is reasonably linear within the range of frequencies of interest.

Example 6.1: A Simple Differentiator



$$v_{in}(t) = A\cos(\omega_c t + \beta \sin(\omega_m t))$$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{1 + j\omega RC}$$

$$\omega << (RC)^{-1} \longrightarrow H(\omega) \approx j\omega RC$$

$$h(t) = RC\frac{d}{dt}$$

Recall:

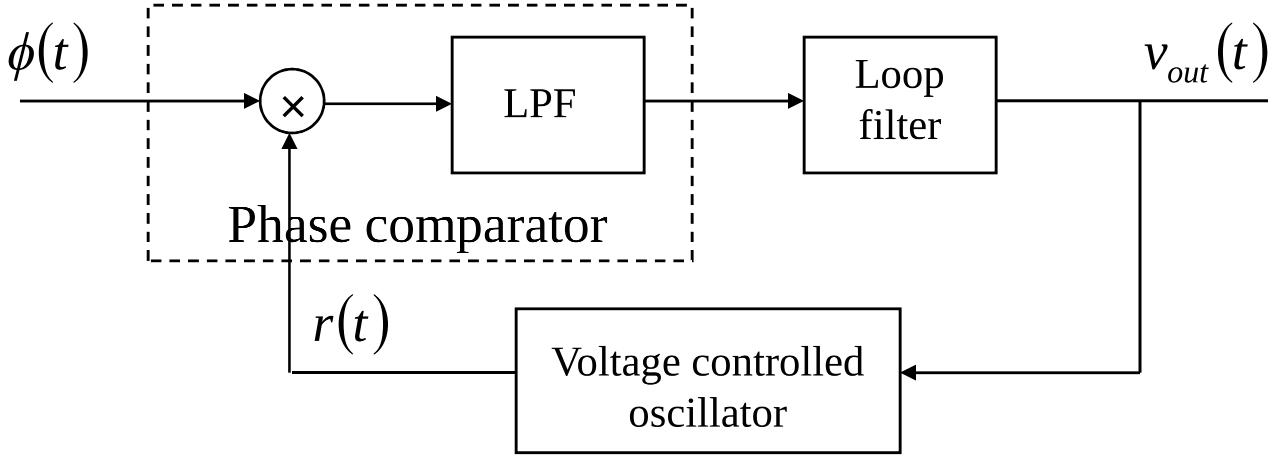
V=IR R: resistor

V=I(1/jwC) C: capacitor

V=I(jwL) L: inductor

$$v_{out} \approx RC\frac{dv_{in}(t)}{dt} = \underbrace{-ARC\omega_c\left[1 + \frac{\beta\omega_m}{\omega_c}\cos\omega_m t\right]}_{\text{envelope}}\sin(\omega_c t + \beta \sin\omega_m t)$$

Indirect Method: Phase Locked Loop (PLL)



Input signal $$\varphi(t) = A_c \cos\left[\omega_c t + \theta_i(t)\right] = A_c \cos\left[\omega_c t + k_i \int_0^t f(\tau)\, d\tau\right]$$

The voltage-controlled oscillator (VCO) produces an instantaneous frequency which is proportional to v_{out}

$$\theta_r = k_r \int_0^t v_{out}(\tau)\, d\tau$$

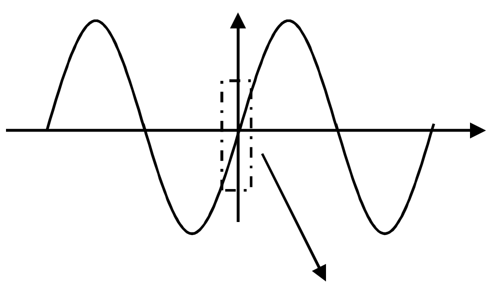
Assume VCO output as $$r(t) = -A_r \sin\left[\omega_c t + k_r \int_0^t v_{out}(\tau)\, d\tau\right] = -A_r \sin\left[\omega_c t + \theta_r(t)\right]$$

PLL (cont'd)

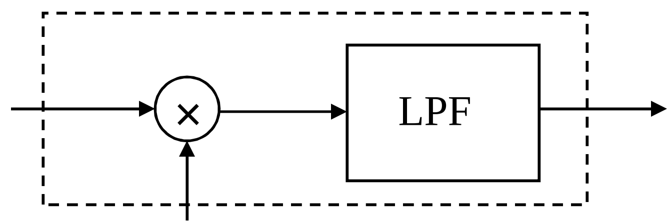
Low pass filter output

$$\phi(t)r(t) = \underbrace{\frac{A_c A_r}{2}\sin[\theta_i(t) - \theta_r(t)]}_{\text{Only this term passes the filter}} - \frac{A_c A_r}{2}\sin[2\omega_c t + \theta_i(t) - \theta_r(t)]$$

For small $(\theta_i(t) - \theta_r(t)) \longrightarrow \sin[\theta_i(t) - \theta_r(t)] \approx \theta_i(t) - \theta_r(t)$



In the linearized region, this scheme can be used as a phase comparator



PLL (cont'd)

--- When the input signal is applied, phase comparison with VCO generates error voltage. In turn, this forces VCO to synchronize itself to the input frequency.

--- In the "lock" position (i.e. VCO is synchronized) VCO frequency becomes identical to input frequency.

--- As the input frequency varies slowly with the message signal, PLL is able to track input frequency through changes in error voltage.

$$\theta_i(t) \approx \theta_r(t)$$

$$k_i \int_0^t f(\tau)d\tau \approx k_r \int_0^t v_{out}(\tau)d\tau \longrightarrow v_{out}(t) \propto f(t)$$

Example 6.2: A 10MHz signal is frequency-modulated by a sinusoidal signal such that the peak frequency deviation is 50kHz. Determine the bandwidth of the FM signal if the frequency of the modulating sinusoid is
a) 500kHz b) 500 Hz c)10kHz.

a) $\beta = \Delta f / f_m = 50 \cdot 10^3 / 500 \cdot 10^3 = 0.1$ Narrowband signal

$B \approx 2 f_m = 1 \text{ MHz}$ $\qquad B = 2 f_m (1+\beta)\big|_{\beta=0.1} = 1.1 \text{ MHz}$

b) $\beta = \Delta f / f_m = 50 \cdot 10^3 / 500 = 100$ Wideband signal

$B \approx 2\Delta f = 100 \text{ kHz}$ $\qquad B = 2 f_m (1+\beta)\big|_{\beta=100} = 101 \text{ kHz}$

c) $\beta = \Delta f / f_m = 50 \cdot 10^3 / 10 \cdot 10^3 = 5$

$B = 2nf_m = 2 \cdot 8 \cdot 10 \cdot 10^3 = 160 \text{ kHz}$ Using definition of “significant sidebands”

$B = 2 f_m (1+\beta)\big|_{\beta=5} = 120 \text{ kHz}$

Example 6.3 Carrier signal $c(t) = 10\cos(\omega_c t)$

Message signal $f(t) = \cos(20\pi t)$

The message is used to frequency modulate the carrier with k_f=100π. Find the expression for the modulated signal and determine how many harmonics should be selected to contain 99% of the modulated signal power.

$$\phi(t) = 10\cos\left(\omega_c t + k_f \int_0^t \cos(20\pi\tau)d\tau\right) = 10\cos(\omega_c t + 5\sin(20\pi t))$$

$$\phi(t) = 10\sum_{n=-\infty}^{+\infty} J_n(5)\cos[(\omega_c + 20\pi n)t]$$

$$P = \overline{\phi^2(t)} = \frac{A_c^2}{2}\sum_{n=-\infty}^{+\infty} J_n^2(\beta) = \frac{A_c^2}{2} = 50$$

$$50\sum_{n=-k}^{k} J_n^2(5) \geq 0.99 \times 50 \longrightarrow 50\left[J_0^2(5) + 2\sum_{n=1}^{k} J_n^2(5)\right] \geq 49.5$$

By trial and error, k=6

Example 6.4 $\phi_{NBFM}(t) = \cos\left(2\pi 10^5 t + 2\pi 25 \int_0^t x_m(\tau)d\tau\right)$

$$\phi_{WBFM}(t) = \cos\left(2\pi 10^8 t + 2\pi 75\ 10^3 \int_0^t x_m(\tau)d\tau\right)$$

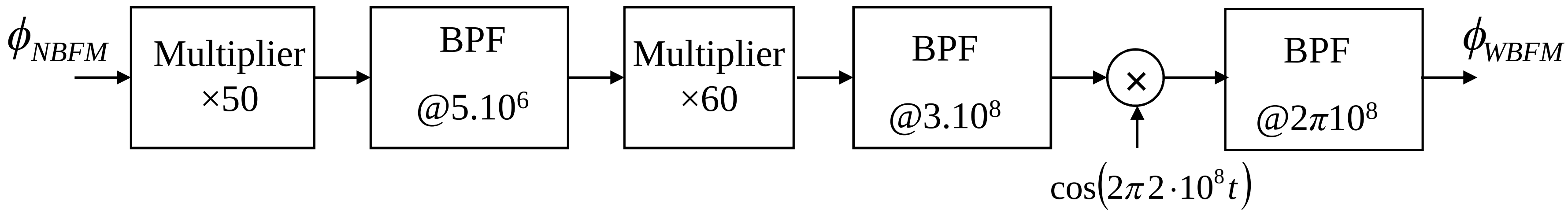
$$\omega_m = 2\pi 10^4$$

$$\max(|x_m(t)|) = 1$$

$$n < \frac{\omega_c - 2\omega_m}{2\Delta\omega} + \frac{1}{2} \approx 1600$$

$$\left.\begin{array}{l}\Delta\omega = 2\pi 25 \\ \Delta\omega' = 2\pi 75 \cdot 10^3\end{array}\right\} \quad \frac{\beta'}{\beta} = \frac{\Delta\omega'}{\Delta\omega} = 3000 > 1600$$

Frequency multiplier can not be implemented with one step!

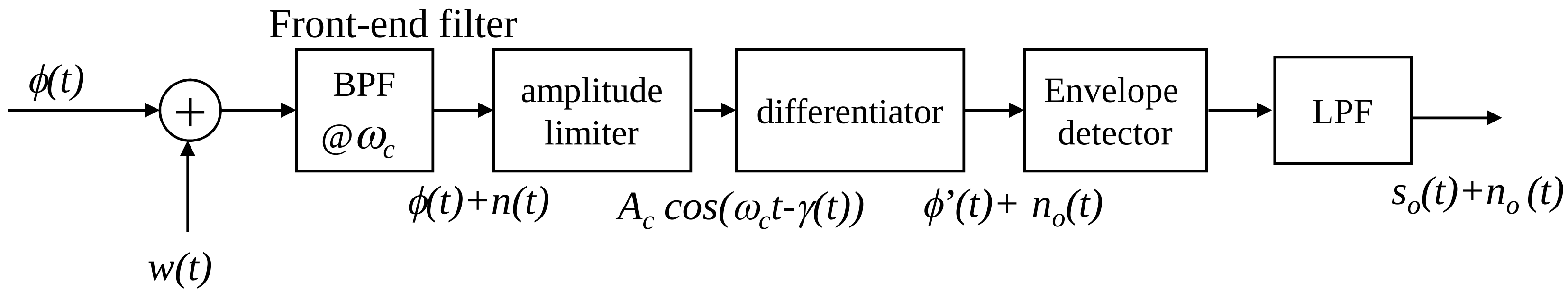


First multiplication $\quad 50 \times 10^5 = 5 \cdot 10^6$

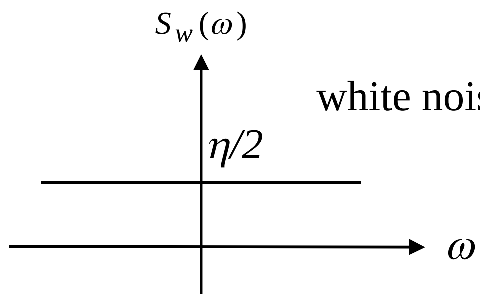
Second multiplication $\quad 60 \times 5 \cdot 10^6 = 3 \cdot 10^8$

$3 \cdot 10^8 \longrightarrow 1 \cdot 10^8$ $\quad$ Carrier frequency needs to be adjusted.

Signal-to-Noise Ratio (SNR) in Frequency Discriminator



Input toBPF:



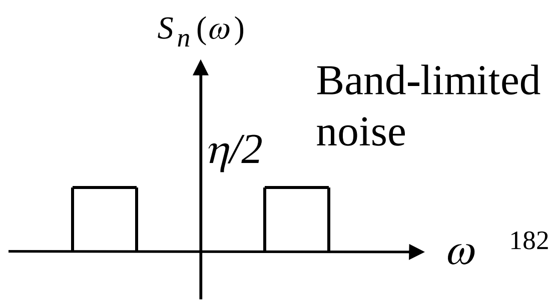
(1) $\phi(t) = A_c \cos[\omega_c t + \underbrace{k_f \int_0^t f(\tau) d\tau}_{\theta(t)}]$

(2) $w(t)$ is white noise with PSD $S_w(\omega) = \eta / 2$ for $-\infty < \omega < \infty$.

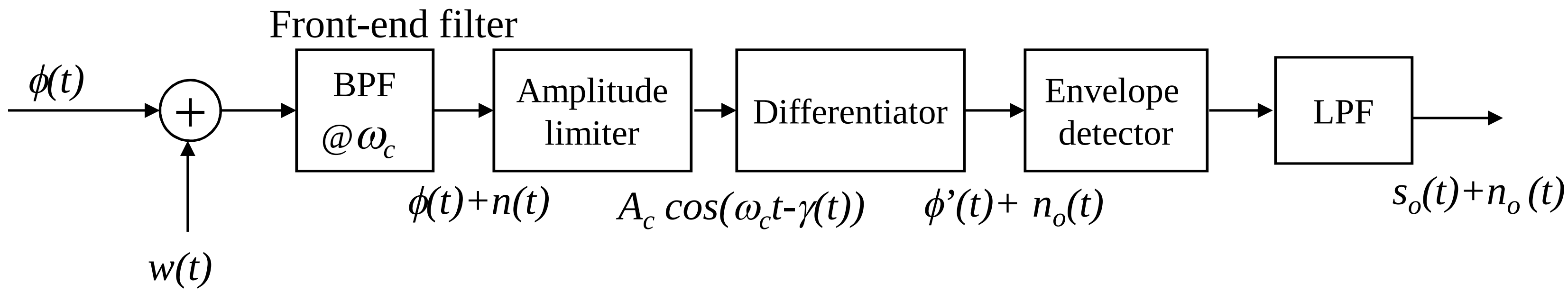
Output signalfrom BPF, whose bandwidth equals the bandwidth (B) of $\phi(t)$:

(1) No change to $\phi(t)$.

(2) $w(t) \rightarrow n(t)$, whose PSD $S_n(\omega) = \begin{cases} \eta / 2, & \text{for } -B < \omega < B \\ 0, & \text{otherwise.} \end{cases}$.



SNR in Frequency Discriminator (cont'd)



Amplitude Limiter removes any undesired envelope variation (i.e. keeping A_c constant) in $\phi(t)+n(t)$

Differentiator output

$$\frac{d\phi(t)}{dt} = -A\left[\omega_c + k_f\, f(t)\right]\sin\left(\omega_c t + k_f \int_0^t f(\tau)d\tau\right)$$

Envelope detector's output (after removing the ω_c term)

Signal term $s_o(t) = k_f\, f(t)$ with power $S_o = \overline{s_o^2(t)} = k_f^2\, \overline{f^2(t)}$

SNR in Frequency Discriminator (cont'd)

How to find the spectral density of noise process?

The band-limited noise $n(t)$ in the presence of an unmodulated carrier $A_c\cos(\omega_c t)$:

$$
\begin{aligned}
A\cos(\omega_c t)+n(t) &= A_c\cos(\omega_c t)+n_c(t)\cos(\omega_c t)-n_s(t)\sin(\omega_c t) \\
&= \left[A_c+n_c(t)\right]\cos(\omega_c t)-n_s(t)\sin(\omega_c t) \\
&= \underbrace{r(t)}_{\substack{\text{amplitude}\\ \text{noise}}}\cos[\omega_c t+\underbrace{\gamma(t)}_{\substack{\text{phase}\\ \text{noise}}}]
\end{aligned}
$$

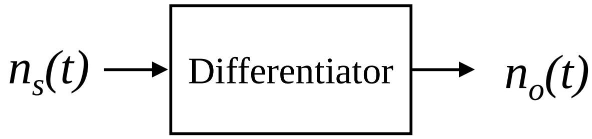
$$
r(t)=\sqrt{\left[A_c+n_c(t)\right]^2+\left[n_s(t)\right]^2} \qquad \gamma(t) = \tan^{-1}\left\{n_s(t)/\left[A_c+n_c(t)\right]\right\}
$$

Since dealing with frequency modulation, ignore amplitude-noise $r(t)$ but focus on the phase-noise:

$$
\gamma(t) = \tan^{-1}\left\{\frac{n_s(t)}{A_c+n_c(t)}\right\} \underset{A_c<<n_c(t)}{\approx} \tan^{-1}\left\{\frac{n_s(t)}{A_c}\right\} \underset{A_c<<n_s(t)}{\approx} \frac{n_s(t)}{A_c}
$$

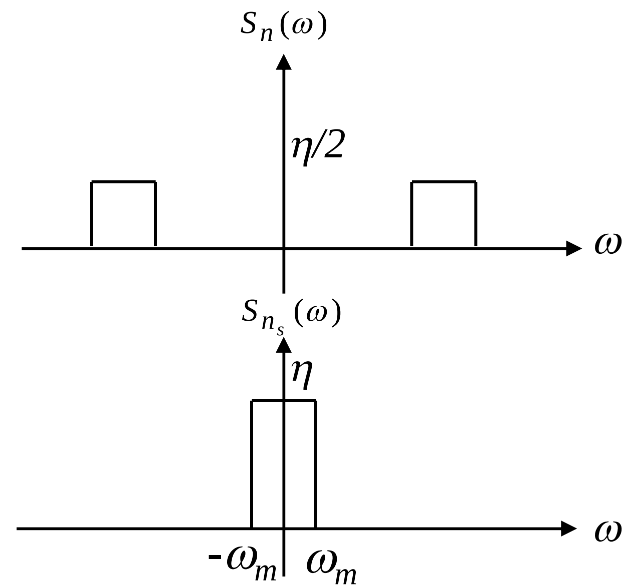
$$
n_o(t) = \frac{d\gamma(t)}{dt} = \frac{1}{A_c}\frac{dn_s(t)}{dt} \qquad S_{n_o}(\omega) = ?
$$

SNR in Frequency Discriminator (cont'd)



$$h(t) = \frac{1}{A_c}\frac{d}{dt} \longleftrightarrow H(\omega) = \frac{1}{A_c} j\omega$$

$$S_{n_o}(\omega) = |H(\omega)|^2 S_{n_s}(\omega) = \frac{\omega^2}{A_c^2} S_{n_s}(\omega)$$



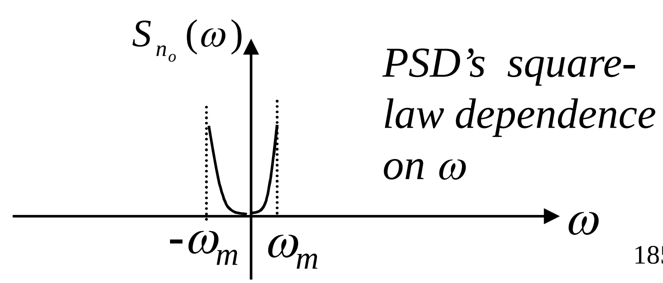
At the LPF's output:

$$S_{n_o}(\omega) = \frac{\omega^2}{A_c^2} S_{n_s}(\omega)$$

See lecture notes on noise

$$= \frac{\omega^2}{A_c^2}\left[S_n(\omega-\omega_c) + S_n(\omega+\omega_c)\right]_{LPF}$$

$$= \frac{\omega^2}{A_c^2}\left[\frac{\eta}{2} + \frac{\eta}{2}\right]_{LPF} = \frac{\omega^2 \eta}{A_c^2}$$



$$\Rightarrow \quad N_0 = \overline{n_o^2(t)} = \frac{\eta}{2\pi A_c^2}\int_{-\omega_m}^{\omega_m} \omega^2 d\omega \quad = \quad \frac{\eta \omega_m^3}{3\pi A_c^2}$$

SNR in Frequency Discriminator (cont'd)

Signalterm $s_o(t) = k_f\, f(t)$ with power $S_o = \overline{s_o^2(t)} = k_f^2\, \overline{f^2(t)}$

Noise term $n_o(t)$ with power N_0

$$\Rightarrow \quad \frac{S_o}{N_o} = \frac{k_f^2\, \overline{f^2(t)}}{\left(\dfrac{\eta\omega_m^3}{3\pi A_c^2}\right)} = \frac{3\pi A_c^2 k_f^2\, \overline{f^2(t)}}{\eta\omega_m^3}$$

$B(\text{rad}) \approx 2(\omega_m + \Delta\omega)$ $\quad$ $\Delta\omega$ is proportional to k_f

For wideband FM, SNR increases with increasing bandwidth through k_f dependence.

If $f(t) = a\cos\omega_m t \;\Rightarrow\; \overline{f^2(t)} = \dfrac{a^2}{2}$

$$\Rightarrow \quad \frac{S_o}{N_o} = \frac{k_f^2\, \frac{1}{2} a^2\, 3\pi A_c^2}{\eta\omega_m^3} = \frac{3\pi A_c^2 (\Delta\omega)^2}{2\eta\omega_m^3} = \frac{3\pi A_c^2 \beta^2}{2\eta\omega_m}$$

Comparison Between FM & DSB-LC

Consider DSB-LC with carrier signal $A_c \cos(\omega_c t)$ and message signal $f(t) = a\cos(\omega_m t)$

$$N_{0(DSB-LC)} = \overline{n_c^2(t)} = \frac{2}{2\pi}\int_{\omega_c-\omega_m}^{\omega_c+\omega_m} \frac{\eta}{2}\, d\omega \qquad \Rightarrow \left(\frac{S_o}{N_o}\right)_{DSB-LC} = \frac{a^2/2}{\eta\omega_m/\pi}$$

$$= \frac{\eta\omega_m}{\pi}$$

For the message signal $f(t) = a\cos\omega_m t$ $\quad \Rightarrow \left(\frac{S_o}{N_o}\right)_{FM} = \frac{3\pi A_c^2 \beta^2}{2\eta\omega_m}$

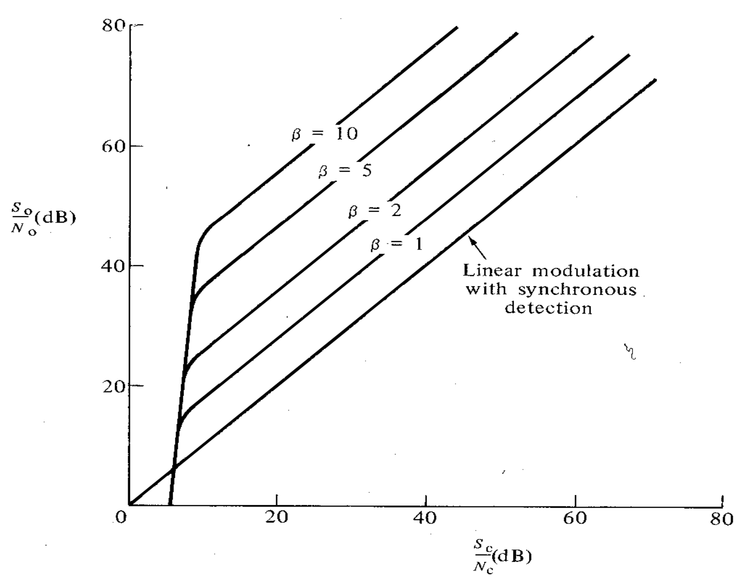
$$\left(\frac{S_o}{N_o}\right)_{FM} = 3\left(\frac{A_c}{a}\right)^2 \beta^2 \left(\frac{S_o}{N_o}\right)_{DSB-LC}$$

Assume unity modulation index, m=1, i.e. A_c=*a*. *This is the most favorable conditions for DSB-LC in terms of power requirements.*

$$\left(\frac{S_o}{N_o}\right)_{FM} = 3\beta^2 \left(\frac{S_o}{N_o}\right)_{DSB-LC} \Rightarrow \text{Iff} \quad \beta > \frac{1}{\sqrt{3}} = 0.599 \quad \text{will} \quad \left(\frac{S_o}{N_o}\right)_{FM} > \left(\frac{S_o}{N_o}\right)_{DSB-LC}$$

Comparison Between FM & AM

The previous argument holds for the comparison between FM and DSB-LC. We have earlier discussed that DSB-LC and SSB has the identical output SNR in the presence of white noise.



$$\left(\frac{S_o}{N_o}\right)_{FM} = 3\beta^2\left(\frac{S_o}{N_o}\right)_{AM}$$

$$B \approx 2\omega_m(1+\beta)$$

FM improves the SNR at the cost of increased bandwidth.

The exchange of bandwidth for SNR in FM can NOT be continued indefinitely. → The noise power increases with the increased bandwidth and results in very poor system performance. This is known as "threshold effect" and the above analysis does not hold anymore.

FM's Phase-Noise's Threshold Effect

(Stremler Chapter 6.9)

Band-limited noise *n(t)* in the presence of an unmodulated carrier $A_c \cos(\omega_c t)$ was previously modeled as:

$$
\begin{aligned}
A\cos(\omega_c t) + n(t) &= A\cos(\omega_c t) + n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \\
&= \underbrace{r(t)}_{\substack{\text{amplitude}\\ \text{noise}}} \cos[\omega_c t + \underbrace{\gamma(t)}_{\substack{\text{phase}\\ \text{noise}}}]
\end{aligned}
$$

Suppose $n_c(t)$ and $n_s(t)$ are approximated as constant with respect to t. Then,

$$
\begin{aligned}
A\cos(\omega_c t) + n(t) &= A\cos(\omega_c t) + n_c \cos(\omega_c t) - n_s \sin(\omega_c t) \\
&= A\cos(\omega_c t) + B\cos(\omega_c t + \theta_b) \\
&= \left[A + B\cos(\theta_b)\right]\cos(\omega_c t) - \left[B\sin(\theta_b)\right]\sin(\omega_c t) \\
&= C\cos(\omega_c t + \theta_c)
\end{aligned}
$$

where

$$
\mathrm{B} = \sqrt{n_c^2 + n_s^2}, \quad \theta_b = \tan^{-1}\left(\frac{n_s}{n_c}\right)
$$

$$
C = \sqrt{A^2 + 2AB\cos\theta_b + B^2}, \quad \theta_c = \tan^{-1}\left(\frac{B\sin\theta_b}{A + \cos\theta_b}\right)
$$

FM's Phase-Noise's Threshold Effect (cont'd)

As frequency modulation concerns primarily the frequency/ phase, the analysis will next focus on θ_c :

$$\theta_c = \tan^{-1}\left\{\frac{B\sin(\theta_b)}{A+B\cos(\theta_b)}\right\} \qquad \overline{\theta_c^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\tan^{-1}\left\{\frac{B\sin(\theta_b)}{A+B\cos(\theta_b)}\right\}\right]^2 d\theta_b$$

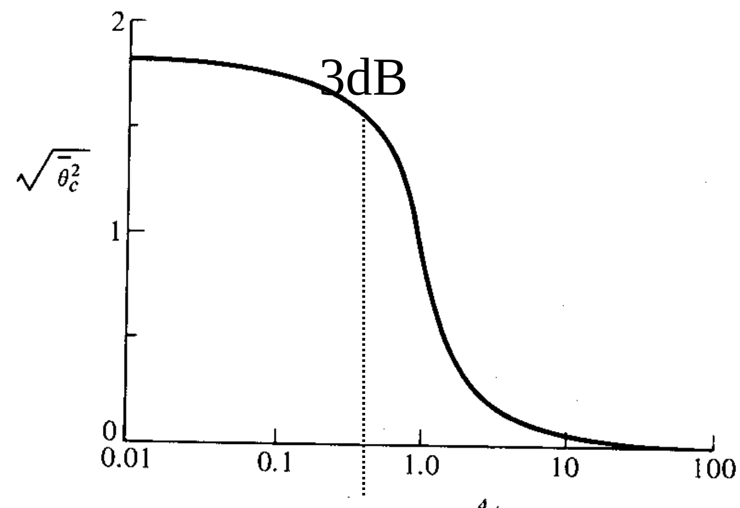
can be numerically evaluated.

n_c and n_s are comparable to A, → *no approximations*

Note A/B can be regarded as a "voltage signal-to-noise ratio".

$$\frac{A}{B} = \frac{A}{\sqrt{n_c^2 + n_s^2}}$$

Mean-square phase noise increases rapidly when the ratio A/B is smaller than about 3dB. This is known as "*threshold effect*".



(b)