CHAPTER ONE

INTRODUCTION TO MEASUREMENTS

1. **Definition.** *Measurement* is the process of determinating the value of a physical quantity experimentally with the help of special technical means called *measuring instruments.*

A measurable quantity (briefly—measurand) is a property of phenomena, bodies, or substances that can be defined qualitatively and expressed quantitatively.

Measurable quantities are also called physical quantities. The principal feature of physical quantities is that they can be measured.

The *value of a physical quantity* is the product of a number and a unit adapted for these quantities. It is found as the result of a measurement.

2. Measurement units

The very first measurement units were those used in barter trade to quantify the amounts being exchanged and to establish clear rules about the relative values of different commodities. Such early systems of measurement were based on whatever was available as a measuring unit. For purposes of measuring length, the human torso was a convenient tool, and gave us units of the hand, the foot and the cubit. Although generally adequate for barter trade systems, such measurement units are of course imprecise, varying as they do from one person to the next. Therefore, there has been a progressive movement towards measurement units that are defined much more accurately.

Table 1.1. and table 1.2. show standards for defining units used for measuring a range of physical variables.

Table 1.1 Definitions of standard units

Physical quantity	*Standard unit*	*Definition*
Length	metre	The length of path travelled by light in an interval of 1/299 792 458 seconds
Mass	kilogram	The mass of a platinum–iridium cylinder kept in the International Bureau of Weights and Measures, Sèvres, Paris
Time	second	9.192631770×10^9 cycles of radiation from vaporized caesium-133 (an accuracy of 1 in 10^{12} or 1 second in 36 000 years)
Temperature	kelvin	The temperature difference between absolute zero and the triple point of water is defined as 273.16 kelvin
Current	ampere	One ampere is the current flowing through two infinitely long parallel conductors of negligible cross-section placed 1 metre apart in a vacuum and producing a force of 2×10^{-7} newtons per metre length of conductor
Luminous intensity	candela	One candela is the luminous intensity in a given direction from a source emitting monochromatic radiation at a frequency of 540 terahertz (Hz $\times 10^{12}$) and with a radiant density in that direction of 1.4641 mW/steradian. (1 steradian is the solid angle which, having its vertex at the centre of a sphere, cuts off an area of the sphere surface equal to that of a square with sides of length equal to the sphere radius)
Matter	mole	The number of atoms in a 0.012 kg mass of carbon-12

Table 1.2 Fundamental and derived SI units

(a) Fundamental units

Quantity	*Standard unit*	*Symbol*
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Matter	mole	mol

(b) Supplementary fundamental units

Quantity	*Standard unit*	*Symbol*
Plane angle	radian	rad
Solid angle	steradian	sr

(c) Derived units

Quantity	*Standard unit*	*Symbol*	*Derivation formula*
Area	square metre	m^2	
Volume	cubic metre	m^3	
Velocity	metre per second	m/s	
Acceleration	metre per second squared	m/s^2	
Angular velocity	radian per second	rad/s	
Angular acceleration	radian per second squared	rad/s^2	
Density	kilogram per cubic metre	kg/m^3	
Specific volume	cubic metre per kilogram	m^3/kg	
Mass flow rate	kilogram per second	kg/s	
Volume flow rate	cubic metre per second	m^3/s	
Force	newton	N	$kg\ m/s^2$
Pressure	newton per square metre	N/m^2	
Torque	newton metre	N m	
Momentum	kilogram metre per second	kg m/s	
Moment of inertia	kilogram metre squared	$kg\ m^2$	
Kinematic viscosity	square metre per second	m^2/s	
Dynamic viscosity	newton second per square metre	$N\ s/m^2$	
Work, energy, heat	joule	J	Nm
Specific energy	joule per cubic metre	J/m^3	
Power	watt	W	J/s
Thermal conductivity	watt per metre kelvin	W/m K	
Electric charge	coulomb	C	A s
Voltage, e.m.f., pot. diff.	volt	V	W/A
Electric field strength	volt per metre	V/m	
Electric resistance	ohm	Ω	V/A
Electric capacitance	farad	F	A s/V
Electric inductance	henry	H	V s/A
Electric conductance	siemen	S	A/V
Resistivity	ohm metre	Ωm	
Permittivity	farad per metre	F/m	
Permeability	henry per metre	H/m	
Current density	ampere per square metre	A/m^2	

(continued overleaf)

Table 1.2 *(continued)*

(c) Derived units

Quantity	*Standard unit*	*Symbol*	*Derivation formula*
Magnetic flux	weber	Wb	V s
Magnetic flux density	tesla	T	Wb/m^2
Magnetic field strength	ampere per metre	A/m	
Frequency	hertz	Hz	s^{-1}
Luminous flux	lumen	lm	cd sr
Luminance	candela per square metre	cd/m^2	
Illumination	lux	lx	lm/m^2
Molar volume	cubic metre per mole	m^3/mol	
Molarity	mole per kilogram	mol/kg	
Molar energy	joule per mole	J/mol	

3. Elements of measuring instrument system.

A *measuring system* exists to provide information about the physical value of some variable being measured. In simple cases, the system can consist of only a single unit that gives an output reading or signal according to the magnitude of the unknown variable applied to it. However, in more complex measurement situations, a measuring system consists of several separate elements as shown in Figure 1. These components might be contained within one or more boxes, and the boxes holding individual measurement elements might be either close together or physically separate. The term *measuring instrument* is commonly used to describe a measurement system.

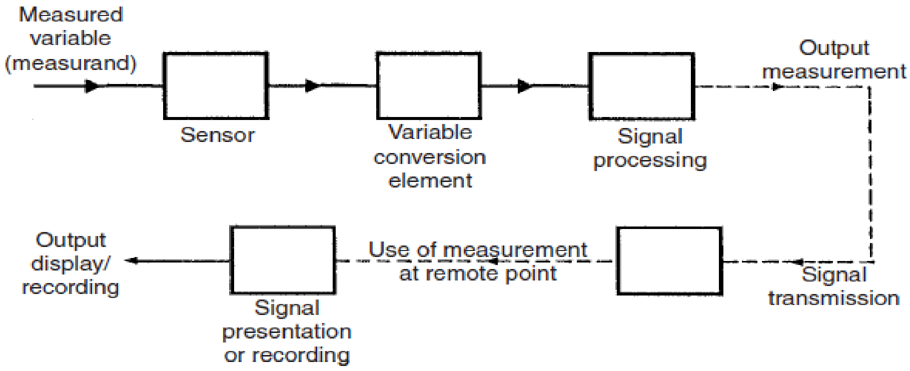

Figure (1) Elements of a measuring instrument.

3.1. The sensor

The sensor gives an output that is a function of the measurand (the input applied to it). For most but not all sensors, this function is at least approximately linear. Some examples of primary sensors are a liquid-in-glass thermometer, a thermocouple and a strain gauge. In the case of the mercury-in-glass thermometer, the output reading is given in terms of the level of the mercury, and so this particular primary sensor is also a complete measurement system in itself. However, in general, the primary sensor is only part of a measurement system.

3.2. Variable conversion

Variable conversion elements are needed where the output variable of a primary transducer is in an inconvenient form and has to be converted to a more convenient form. For instance, the displacement-measuring strain gauge has an output in the form of a varying resistance. The resistance change cannot be easily measured and so it is converted to a change in voltage by a *bridge circuit*, which is a typical example of a variable conversion element. In some cases, the primary sensor and variable conversion element are combined, and the combination is known as a *transducer*.

3.3. Signal processing

Signal processing elements exist to improve the quality of the output of a measurement system in some way. A very common type of signal processing element is the electronic amplifier, which amplifies the output of the primary transducer or variable conversion element, thus improving the sensitivity and resolution of measurement. This element of a measuring system is particularly important where the primary transducer has a low output. For example, thermocouples have a typical output of only a few millivolts. Other types of signal processing element are those that filter out induced noise and remove mean levels etc. In some devices, signal processing is incorporated into a transducer, which is then known as a *transmitter*.

- In addition to these three components just mentioned, some measurement systems have one or two other components, firstly to transmit the signal to some remote point and secondly to display or record the signal if it is not fed automatically into a feedback control system. Signal transmission is needed when the observation or application point of the output of a measurement system is some distance away from the site of the primary transducer. Sometimes, this separation is made solely for purposes of convenience, but more often it follows from the physical inaccessibility or environmental unsuitability of the site of the primary transducer for mounting the signal presentation/recording unit. The signal transmission element has traditionally consisted of single or multi-cored cable, which is often screened to minimize signal corruption by induced electrical noise. However, fibre-optic cables are being used in ever increasing numbers in modern installations, in part because of their low transmission loss and imperviousness to the effects of electrical and magnetic fields.
- The final optional element in a measurement system is the point where the measured signal is utilized. In some cases, this element is omitted altogether because the measurement is used as part of an automatic control scheme, and the transmitted signal is fed directly into the control system. In other cases, this element in the measurement system takes the form either of a signal presentation unit or of a signal-recording unit.

4. ERRORS

4.1. Definition :

If A is the true value of the measurable quantity and A' is the result of measurement, then the absolute error of measurement is $\zeta = A' - A$. This equation is often used as a definition of this term, but by doing that, one narrows the essence of this term.

The error expressed in absolute form is called the ***absolute measurement error*** (ζ).
The error expressed in relative form is called the ***relative measurement error***.

The relative error is the error expressed as a fraction of the true value of the measurable quantity $\varepsilon = (A' - A)/A$. Relative errors are normally given as percent and sometimes per thousand (denoted by ‰). Very small errors, which are encountered in the most precise measurements, are customarily expressed directly as fractions of the measured quantity.

4.2. types of errors:

4.2.1. Systematic errors

A measurement error is said to be ***systematic*** if it remains constant with repeated measurements or changes in a regular fashion in repeated measurements of one and the same quantity. The observed and estimated systematic error is eliminated from measurements by introducing corrections. However, it is impossible to eliminate completely the systematic error in this manner. Some part of the error will remain, and then this residual error will be the systematic component of the measurement error.

They can be divided into two basic groups: **instrumental errors and environmental errors.** Instrumental errors are inherent within the instrument, arising because of mechanical structures, electronic designs, improper adjustments, wrong applications, and so on. They can also be sub classified as loading, scale, zero, and response time errors. Environmental errors are caused by environmental factors such as temperature and humidity.

4.2.2. Sources of systematic errors:

4.2.2.1. System disturbance due to measurement

Disturbance of the measured system by the act of measurement is a common source of systematic error. If we were to start with a beaker of hot water and wished to measure its temperature with a mercury-in-glass thermometer, then we would take the thermometer, which would initially be at room temperature, and plunge it into the water. In so doing, we would be introducing a relatively cold mass (the thermometer) into the hot water and a heat transfer would take place between the water and the thermometer. This heat transfer would lower the temperature of the water. Whilst the reduction in temperature in this case would be so small as to be undetectable by the limited measurement resolution of such a thermometer, the effect is finite and clearly establishes the principle that, in nearly all measurement situations, the process of measurement disturbs the system and alters the values of the physical quantities being measured.

4.2.2.2. Errors due to environmental inputs

An environmental input is defined as an apparently real input to a measurement system that is actually caused by a change in the environmental conditions surrounding the measurement system. The fact that the static and dynamic characteristics specified for measuring instruments are only valid for particular environmental conditions (e.g. of temperature and pressure) These specified conditions must be reproduced as closely as possible during calibration exercises because, away from the specified calibration conditions, the characteristics of measuring instruments vary to some extent and cause measurement errors.

The magnitude of this environment-induced variation is quantified by the two constants known as **sensitivity drift and zero drift,** both of which are generally included in the published specifications for an instrument. Such variations of environmental conditions away from the calibration conditions are sometimes described as modifying inputs to the measurement system because they modify the output of the system. When such modifying inputs are present, it is often difficult to determine how much of the output change in a measurement system is due to a change in the measured variable and how much is due to a change in environmental conditions.

4.2.2.3. Wear in instrument components

Systematic errors can frequently develop over a period of time because of wear in instrument components. Recalibration often provides a full solution to this problem.

4.2.2.4. Connecting leads

In connecting together the components of a measurement system, a common source of error is the failure to take proper account of the resistance of connecting leads (or pipes in the case of pneumatically or hydraulically actuated measurement systems). For instance, in typical applications of a resistance thermometer, it is common to find that the thermometer is separated from other parts of the measurement system by perhaps 100 metres. The resistance of such a length of 20 gauge copper wire is 7 Ω , and there is a further complication that such wire has a temperature coefficient of 1mΩ/ºC.

Therefore, careful consideration needs to be given to the choice of connecting leads. Not only should they be of adequate cross-section so that their resistance is minimized, but they should be adequately screened if they are thought likely to be subject to electrical or magnetic fields that could otherwise cause induced noise. Where screening is thought essential, then the routing of cables also needs careful planning. However, by changing the route of the cables between the transducers and the control room, the magnitude of this induced noise was reduced by a factor of about ten.

4.2.3. Reduction of systematic errors

The prerequisite for the reduction of systematic errors is a complete analysis of the measurement system that identifies all sources of error. Simple faults within a system, such as bent meter needles and poor cabling practices, can usually be readily and cheaply rectified once they have been identified. However, other error sources require more detailed analysis and treatment. Various approaches to error reduction are considered below.

4.2.3.1 Careful instrument design

Careful instrument design is the most useful weapon in the battle against environmental inputs, by reducing the sensitivity of an instrument to environmental inputs to as low a level as possible. For instance, in the design of strain gauges, the element should be constructed from a material whose resistance has a very low temperature coefficient (i.e. the variation of the resistance with temperature is very small). However, errors due to the way in which an instrument is designed are not always easy to correct, and a choice often has to be made between the high cost of redesign and the alternative of accepting the reduced measurement accuracy if redesign is not undertaken.

4.2.3.2 Method of opposing inputs

The method of opposing inputs compensates for the effect of an environmental input in a measurement system by introducing an equal and opposite environmental input that cancels it out. One example of how this technique is applied is in the type of millivoltmeter. This consists of a coil suspended in a fixed magnetic field produced by a permanent magnet. When an unknown voltage is applied to the coil, the magnetic field due to the current interacts with the fixed field and causes the coil (and a pointer attached to the coil) to turn. If the coil resistance Rcoil is sensitive to temperature, then any environmental input to the system in the form of a temperature change will alter the value of the coil current for a given applied voltage and so alter the pointer output reading. Compensation for this is made by introducing a compensating resistance Rcomp into the circuit, where Rcomp has a temperature coefficient that is equal in magnitude but opposite in sign to that of the coil. Thus, in response to an increase in temperature, Rcoil increases but Rcomp decreases, and so the total resistance remains approximately the same.

4.2.3.3 Calibration

Instrument calibration is a very important consideration in measurement systems. All instruments suffer drift in their characteristics, and the rate at which this happens depends on many factors, such as the environmental conditions in which instruments are used and the frequency of their use. Thus, errors due to instruments being out of calibration can usually be rectified by increasing the frequency of recalibration.

4.2.3.4. Manual correction of output reading

In the case of errors that are due either to system disturbance during the act of measurement or due to environmental changes, a good measurement technician can substantially reduce errors at the output of a measurement system by calculating the effect of such systematic errors and making appropriate correction to the instrument readings. This is not necessarily an easy task, and requires all disturbances in the measurement system to be quantified. This procedure is carried out automatically by intelligent instruments.

4.2.3.5. Intelligent instruments

Intelligent instruments contain extra sensors that measure the value of environmental inputs and automatically compensate the value of the output reading. They have the ability to deal very effectively with systematic errors in measurement systems, and errors can be attenuated to very low levels in many cases.

4.2.4. Random errors

To define a random measurement errors, imagine that some quantity is measured several times. If there are differences between the results of separate measurements, and these differences cannot be predicted individually and any regularities inherent to them are manifested only in many results, then the error from this scatter of the results is called the ***random error*.**

Random errors are discovered by performing measurements of one and the same quantity repeatedly under the same conditions, whereas systematic errors can be discovered experimentally either by comparing a given result with a measurement of the same quantity performed by a different method or by using a more accurate measuring instrument. However, systematic errors are normally estimated by theoretical analysis of the measurement conditions, based on the known properties of a measurand and of measuring instruments.

The quality of measurements that reflects the closeness of the results of measurements of the same quantity performed under the same conditions is called the ***repeatability of measurements*.** Good repeatability indicates that the random errors are small.

On the other hands, the quality of measurements that reflects the closeness of the results of measurements of the same quantity performed under different conditions, i.e., in different laboratories (at different locations) and using different equipment, is called the ***reproducibility of measurements***. Good reproducibility indicates that both the random and systematic errors are small.

Random errors in measurements are caused by unpredictable variations in the measurement system. They are usually observed as small perturbations of the measurement either side of the correct value, i.e. positive errors and negative errors occur in approximately equal numbers for a series of measurements made of the same constant quantity.

Therefore, random **errors can largely be eliminated by calculating the average of a number of repeated measurements**, provided that the measured quantity remains constant during the process of taking the repeated measurements. This averaging process of repeated measurements can be done automatically by intelligent instruments. The degree of confidence in the calculated mean/median values can be quantified by calculating the standard deviation or variance of the data, these being parameters that describe how the measurements are distributed about the mean value/median.

4.2.5. Statistical analysis of measurements subject to random errors

4.2.5.1. Mean and median values

The average value of a set of measurements of a constant quantity can be expressed as either the mean value or the median value. As the number of measurements increases, the difference between the mean value and median values becomes very small. However, for any set of **n** measurements x_1, x_2, ……, x_n of a constant quantity, the most likely true value is the ***mean*** given by:

This is valid for all data sets where the measurement errors are distributed equally about the zero error value, i.e. where the positive errors are balanced in quantity and magnitude by the negative errors.

The ***median*** is an approximation to the mean that can be written down without having to sum the measurements. The median is the middle value when the measurements in the data set are written down in ascending order of magnitude. For a set of **n** measurements x_1, x_2, …., x_n of a constant quantity, written down in ascending order of magnitude, the median value is given by:

Thus, for a set of 9 measurements x_1, x_2, …, x9 arranged in order of magnitude, the median value is x5. For an even number of measurements, the median value is midway between the two centre values, i.e. for 10 measurements x_1, x_2, …, x_{10}, the median value is given by:

$$(x_5 + x_6)/2.$$

Suppose that the length of a steel bar is measured by a number of different observers and the following set of 11 measurements are recorded (units mm). We will call this measurement set A.

398, 420, 394, 416, 404, 408, 400, 420, 396, 413, 430 Measurement set A

The mean is 409.0 and the median 408. Suppose now that the measurements are taken again using a better measuring rule, and with the observers taking more care, to produce the following measurement set B:

409, 406, 402, 407, 405, 404, 407, 404, 407, 407, 408 Measurement set B

For these measurements, the mean is 406.0 and the median 407. Which of the two measurement sets A and B, and the corresponding mean and median values, should we have most confidence in? Intuitively, we can regard measurement set B as being more reliable since the measurements are much closer together. In set A, the spread between the smallest (396) and largest (430) value is 34, whilst in set B, the spread is only 6.

- ***Thus, the smaller the spread of the measurements, the more confidence we have in the mean or median value calculated.***

Let us now see what happens if we increase the number of measurements by extending measurement set B to 23 measurements. We will call this measurement set C.

409, 406, 402, 407, 405, 404, 407, 404, 407, 407, 408, 406, 410, 406, 405, 408, 406, 409, 406, 405, 409, 406, 407 Measurement set C

Now, the mean is 406.5 and the median D 406.

- ***This confirms our earlier statement that the median value tends towards the mean value as the number of measurements increases.***

4.2.5.2. Standard deviation and variance

Expressing the spread of measurements simply as the range between the largest and smallest value is not in fact a very good way of examining how the measurement values are distributed about the mean value. A much better way of expressing the distribution is to calculate the variance or standard deviation of the measurements. The starting point for calculating these parameters is to calculate the deviation (error) $\mathbf{d_i}$ of each measurement $\mathbf{x_i}$ from the mean value x_{mean}:

$$d_i = x_i - x_{mean}$$

The ***variance* (V)** is then given by :

The *standard deviation* (σ) is simply the square root of the variance. Thus:

<u>Example (1)</u>

Calculate (σ) and V for measurement sets A, B and C above.

o

Solution

First, draw a table of measurements and deviations for set A (mean = 409 as calculated earlier):

Measurement	398	420	394	416	404	408	400	420	396	413	430
Deviation from mean	−11	+11	−15	+7	−5	−1	−9	+11	−13	+4	+21
$(\text{deviations})^2$	121	121	225	49	25	1	81	121	169	16	441

$\sum(\text{deviations}^2) = 1370$; n = number of measurements = 11.

Then, $V = \sum(\text{deviations}^2)/n - 1; = 1370/10 = 137; \sigma = \sqrt{V} = 11.7.$

The measurements and deviations for set B are (mean = 406 as calculated earlier):

Measurement	409	406	402	407	405	404	407	404	407	407	408
Deviation from mean	+3	0	−4	+1	−1	−2	+1	−2	+1	+1	+2
$(\text{deviations})^2$	9	0	16	1	1	4	1	4	1	1	4

From this data, $V = 4.2$ and $\sigma = 2.05$.

The measurements and deviations for set C are (mean = 406.5 as calculated earlier):

Measurement	409	406	402	407	405	404	407	404
Deviation from mean	+2.5	−0.5	−4.5	+0.5	−1.5	−2.5	+0.5	−2.5
$(\text{deviations})^2$	6.25	0.25	20.25	0.25	2.25	6.25	0.25	6.25

Measurement	407	407	408	406	410	406	405	408
Deviation from mean	+0.5	+0.5	+1.5	−0.5	+3.5	−0.5	−1.5	+1.5
$(\text{deviations})^2$	0.25	0.25	2.25	0.25	12.25	0.25	2.25	2.25

Measurement	406	409	406	405	409	406	407
Deviation from mean	−0.5	+2.5	−0.5	−1.5	+2.5	−0.5	+0.5
$(\text{deviations})^2$	0.25	6.25	0.25	2.25	6.25	0.25	0.25

From this data, $V = 3.53$ and $\sigma = 1.88$.

Note that the smaller values of V and σ for measurement set B compared with A correspond with the respective size of the spread in the range between maximum and minimum values for the two sets.

- *Thus, as V and* (σ) *decrease for a measurement set, we are able to express greater confidence that the calculated mean or median value is close to the true value, i.e. that the averaging process has reduced the random error value close to zero.*

- *Comparing V and (σ) for measurement sets B and C, V and (σ) get smaller as the number of measurements increases, confirming that confidence in the mean value increases as the number of measurements increases.*

We have observed so far that random errors can be reduced by taking the average (mean or median) of a number of measurements. However, although the mean or median value is close to the true value, it would only become exactly equal to the true value if we could average an infinite number of measurements. As we can only make a finite number of measurements in a practical situation, the average value will still have some error. This error can be quantified as the *standard error of the mean*, which will be discussed in detail a little later. However, before that, the subject of graphical analysis of random measurement errors needs to be covered.

4.2.6. Graphical data analysis techniques – frequency distributions

Graphical techniques are a very useful way of analyzing the way in which random measurement errors are distributed. The simplest way of doing this is to draw a *histogram*, in which bands of equal width across the range of measurement values are defined and the number of measurements within each band is counted. Figure (2) shows a histogram for set C of the length measurement data given above, in which the bands chosen are 2mm wide. For instance, there are 11 measurements in the range between 405.5 and 407.5 and so the height of the histogram for this range is 11 units. Also, there are 5 measurements in the range from 407.5 to 409.5 and so the height of the histogram over this range is 5 units. The rest of the histogram is completed in a similar fashion. (N.B. The scaling of the bands was deliberately chosen so that no measurements fell on the boundary between different bands and caused ambiguity about which band to put them in.) Such a histogram has the characteristic shape shown by truly random data, with symmetry about the mean value of the measurements.

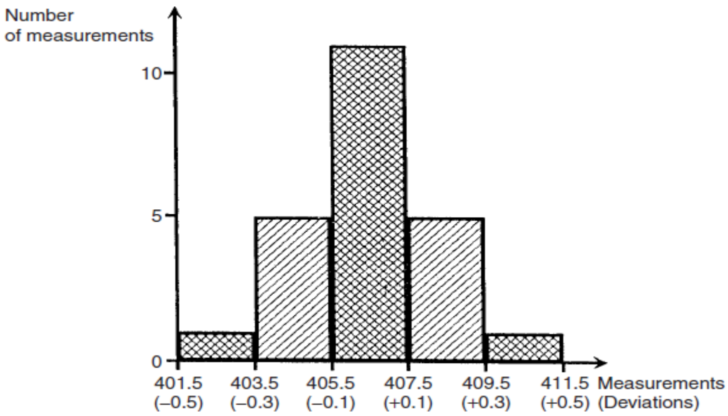

Figure (2) Histogram of measurements and deviations.

As it is the actual value of measurement error that is usually of most concern, it is often more useful to draw a histogram of the deviations of the measurements from the mean value rather than to draw a histogram of the measurements themselves.
The starting point for this is to calculate the deviation of each measurement away from the calculated mean value.

Then a *histogram of deviations* can be drawn by defining deviation bands of equal width and counting the number of deviation values in each band. This histogram has exactly the same shape as the histogram of the raw measurements except that the scaling of the horizontal axis has to be redefined in terms of the deviation values (these units are shown in brackets on figure 2).

Let us now explore what happens to the histogram of deviations as the number of measurements increases. As the number of measurements increases, smaller bands can be defined for the histogram, which retains its basic shape but then consists of a larger number of smaller steps on each side of the peak. In the limit, as the number of measurements approaches infinity, the histogram becomes a smooth curve known as a *frequency distribution curve* as shown in Figure 3.6. The ordinate of this curve is the frequency of occurrence of each deviation value, F(D), and the abscissa is the magnitude of deviation, D.

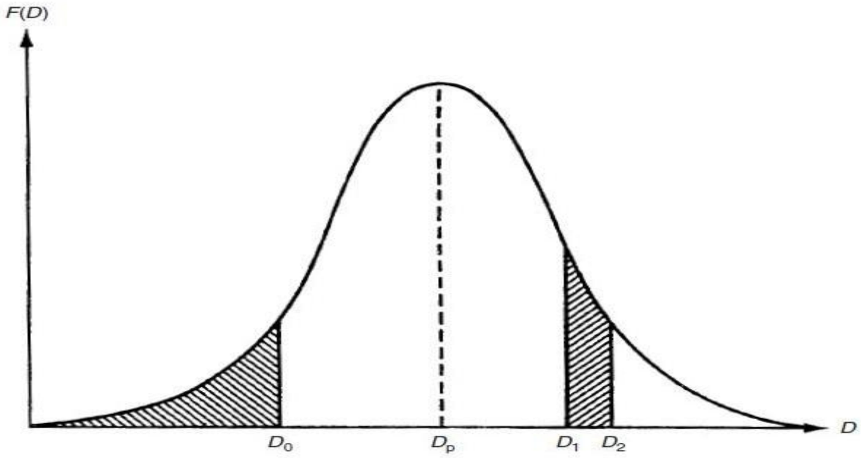

Figure (3) Frequency distribution curve of deviation.

<u>The symmetry of Figures 2 and 3 about the zero deviation value is very useful for showing graphically that the measurement data only has random errors</u>. Although these **<u>figures cannot easily be used to quantify the magnitude and distribution of the errors</u>**, very similar graphical techniques do achieve this. If the height of the frequency distribution curve is normalized such that the area under it is unity, then the curve in this form is known as a *probability curve*, and the height F(D) at any particular deviation magnitude D is known as the ***<u>probability density function</u>* <u>(p.d.f.).</u>** The condition that the area under the curve is unity can be expressed mathematically as:

…………………. (1)

The probability that the error in any one particular measurement lies between two levels D1 and D2 can be calculated by measuring the area under the curve contained between two vertical lines drawn through D1 and D2, as shown by the right-hand hatched area in Figure 3. This can be expressed mathematically as:

Of particular importance for assessing the maximum error likely in any one measurement is the ***<u>cumulative distribution function</u>* <u>(c.d.f.).</u>** This is defined as the probability of observing a value less than or equal to D0, and is expressed mathematically as:

……………………. (3)

<u>Thus, the c.d.f. is the area under the curve to the left of a vertical line drawn through D0, as shown by the left-hand hatched area on Figure 3. The deviation magnitude Dp corresponding with the peak of the frequency distribution curve (Figure 3) is the value of deviation that has the greatest probability. If the errors are entirely random in nature, then the value of Dp will equal zero. Any non-zero value of Dp indicates systematic errors in the data, in the form of a bias that is often removable by recalibration.</u>

<u>Gaussian distribution</u>

Measurement sets that only contain random errors usually conform to a distribution with a particular shape that is called *Gaussian*, although this conformance must always be tested. The shape of a Gaussian curve is such that the frequency of small deviations from the mean value is much greater than the frequency of large deviations. This coincides with the usual expectation in measurements subject to random errors that the number of measurements with a small error is much larger than the number of measurements with a large error. **<u>Alternative names for the Gaussian distribution are the *Normal distribution* or *Bell-shaped distribution*.</u>**

A Gaussian curve is formally defined as a normalized frequency distribution that is symmetrical about the line of zero error and in which the frequency and magnitude of quantities are related by the expression:

………………… (4)

where m is the mean value of the data set x and the other quantities are as defined before.

The last equation is particularly useful for analyzing a Gaussian set of measurements and predicting how many measurements lie within some particular defined range. If the measurement deviations D are calculated for all measurements such that D = x – m , then the curve of deviation frequency F(D) plotted against deviation magnitude D is a Gaussian curve known as the ***error frequency distribution curve***. The mathematical relationship between F(D) and D can then be derived by modifying the last equation to give:

…………………… (5)

The shape of a Gaussian curve is strongly influenced by the value of σ with the width of the curve decreasing as σ becomes smaller. As a smaller σ corresponds with the typical deviations of the measurements from the mean value becoming smaller, this confirms the earlier observation that the mean value of a set of measurements gets closer to the true value as σ decreases.

If the standard deviation is used as a unit of error, the Gaussian curve can be used to determine the probability that the deviation in any particular measurement in a Gaussian data set is greater than a certain value. By substituting the expression for F(D) in equation (5) into the probability equation (2), the probability that the error lies in a band between error levels D1 and D2 can be expressed as:

Solution of this expression is simplified by the substitution:

$$z = D / \sigma \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

The effect of this is to change the error distribution curve into a new Gaussian distribution that has a standard deviation of one (σ = 1) and a mean of zero. This new form, shown in figure (4), is known as a ***standard Gaussian curve***, and the dependent variable is now z instead of D. Equation (6) can now be re-expressed as:

Unfortunately, neither equation (6) nor (8) can be solved analytically using tables of standard integrals, and numerical integration provides the only method of solution. However, in practice, the tedium of numerical integration can be avoided when analyzing data because the standard form of equation (3.15), and its independence from the particular values of the mean and standard deviation of the data, means that
standard Gaussian tables that tabulate F(z) for various values of z can be used.

Standard Gaussian tables

A standard Gaussian table, such as that shown in Table 2, tabulates F(z) for various values of z, where F(z) is given by

Thus, F(z) gives the proportion of data values that are less than or equal to z. This proportion is the area under the curve of F(z) against z that is to the left of z. Therefore, the expression given in (8) has to be evaluated as $[F(z_2) - F(z_1)]$. Study of Table 1.3 shows that F(z) = 0.5 for z = 0. This confirms that, as expected, the number of data values ≤ 0 is 50% of the total. ***This must be so if the data only has random errors.*** It will also be observed that Table 1.3, in common with most published standard Gaussian tables, only gives F(z) for positive values of z. For negative values of z, we can make use of the following relationship because the frequency distribution curve is normalized:

$$F(-z) = 1 - F(z) \quad \ldots\ldots\ldots\ldots \quad (10)$$

F(-z) is the area under the curve to the left of (-z), i.e it represents the proportion of data values ≤-z.

Example (2)

How many measurements in a data set subject to random errors lie outside deviation boundaries of +σ and -σ , i.e. how many measurements have a deviation greater than $|\sigma|$.

Solution

The required number is represented by the sum of the two shaded areas in Figure (4).
This can be expressed mathematically as:

$$P(E < -\sigma \text{ or } E > +\sigma) = P(E < -\sigma) + P(E > +\sigma)$$

For E = - σ , z = -1.0 (from equation (5)) .

Using Table 1.3

$$P(E < -\sigma) = F(-1) = 1 - F(1) = 1 - 0.8413 = 0.1587$$

Similarly, for E = + σ , z= +1.0, table 1.3 gives :

P (E > + σ) = 1- P (< + σ) = 1 – F(1) =1 – 0.8413 = 0.1587

(This last step is valid because the frequency distribution curve is normalized such that the total area under it is unity.)

Thus

P [E < - σ] + P [E > + σ] = 0.1587 + 0.1587 = 0.3174 ≈ 32 %

i.e. 32% of the measurements lie outside the ±σ boundaries, then 68% of the measurements lie inside.

Table (1.3) Standard Gaussian table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
					F(z)					
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9926	0.9928	0.9930	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9986	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

The above analysis shows that, for Gaussian-distributed data values, 68% of the measurements have deviations that lie within the bounds of ±σ. Similar analysis shows that boundaries of ±2σ contain 95.4% of data points, and extending the boundaries to ±3σ encompasses 99.7% of data points. The probability of any data point lying outside particular deviation boundaries can therefore be expressed by the following table.

Deviation boundary	% of data points within boundary	Probability of any particular data point being outside boundary
$\pm\sigma$	68.0	32.0 %
$\pm 2\sigma$	95.40	4.60 %
$\pm 3\sigma$	99.70	0.30 %

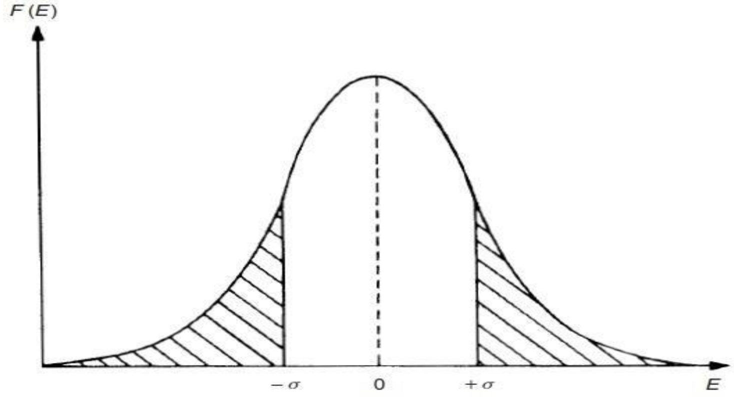

Figure (4) ±σ boundaries.

Standard error of the mean

The foregoing analysis has examined the way in which measurements with random errors are distributed about the mean value. However, we have already observed that some error remains between the mean value of a set of measurements and the true value, i.e. averaging a number of measurements will only yield the true value if the number of measurements is infinite. If several subsets are taken from an infinite data population, then, by the central limit theorem, the means of the subsets will be distributed about the mean of the infinite data set. The error between the mean of a finite data set and the true measurement value (mean of the infinite data set) is defined as the *standard error of the mean*, ˛. This is calculated as:

$$\alpha = \sigma / \sqrt{n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

α tends towards zero as the number of measurements in the data set expands towards infinity. The measurement value obtained from a set of **n** measurements, x1, x2, ……, xn, can then be expressed as:

$$x = x_{mean} \pm \alpha$$

For the data set C of length measurements used earlier, n = 23, σ = 1.88 and ˛ α = 0.39. The length can therefore be expressed as 406.5 ± 0.4 (68% confidence limit). However, it is more usual to express measurements with 95% confidence limits (± 2 σ2 boundaries). In this case, 2 σ = 3.76, 2 α = 0.78 and the length can be expressed as 406.5 ± 0.8 (95% confidence limits).

5. Choosing appropriate instrument

The starting point in choosing the most suitable instrument to use for measurement of a particular quantity in a manufacturing plant or other system is the specification of **the instrument characteristics required, specially parameters like the desired measurement accuracy, resolution, sensitivity and dynamic performance**. It is also essential to **know the environmental conditions that the instrument will be subjected to**, as some conditions will immediately either eliminate the possibility of using certain types of instrument or else will create a requirement for expensive protection of the instrument. **It should also be noted that protection reduces the**

performance of some instruments, especially in terms of their dynamic characteristics (for example, sheaths protecting thermocouples and resistance thermometers reduce their speed of response). Provision of this type of information usually requires the expert knowledge of personnel who are intimately acquainted with the operation of the manufacturing plant or system in question. **Then, a skilled instrument engineer**, having knowledge of all the instruments that are available for measuring the quantity in question, will be able to evaluate the possible list of instruments in terms of their accuracy, cost and suitability for the environmental conditions and thus choose the most appropriate instrument. **As far as possible, measurement systems and instruments should be chosen that are as insensitive as possible to the operating environment**, although this requirement is often difficult to meet because of cost and other performance considerations. The extent to which the measured system will be disturbed during the measuring process is another important factor in instrument choice. For example, significant pressure loss can be caused to the measured system in some techniques of flow measurement.

Published literature is of considerable help in the choice of a suitable instrument for a particular measurement situation. Many books are available that give valuable assistance in the necessary evaluation by providing lists and data about all the instruments available for measuring a range of physical quantities.

However, new techniques and instruments are being developed all the time, and therefore a good instrumentation engineer must keep abreast of the latest developments by reading the appropriate technical journals regularly.

The instrument characteristics discussed in the next chapter are the features that form the technical basis for a comparison between the relative merits of different instruments. **Generally, the better the characteristics, the higher the cost.**

However, in comparing the cost and relative suitability of different instruments for a particular measurement situation, considerations of durability, maintainability and constancy of performance are also very important because the instrument chosen will often have to be capable of operating for long periods without performance degradation and a requirement for costly maintenance. In consequence of this, the initial cost of an instrument often has a low weighting in the evaluation exercise.

Cost is very strongly correlated with the performance of an instrument, as measured by its static characteristics. Increasing the accuracy or resolution of an instrument, for example, can only be done at a penalty of increasing its manufacturing cost. Instrument choice therefore proceeds by specifying the minimum characteristics required by a measurement situation and then searching manufacturers' catalogues to find an instrument whose characteristics match those required. To select an instrument with characteristics superior to those required would only mean paying more than necessary for a level of performance greater than that needed.

Combined effect of systematic and random errors

If a measurement is affected by both systematic and random errors that are quantified as $\pm x$ (systematic errors) and $\pm y$ (random errors), some means of expressing the combined effect of both types of error is needed. One way of expressing the combined error would be to sum the two separate components of error, i.e. to say that the total possible error is $e = \pm(x + y)$. However, a more usual course of action is to express the likely maximum error as follows:

It can be shown (ANSI/ASME, 1985) that this is the best expression for the error statistically, since it takes account of the reasonable assumption that the systematic and random errors are independent and so are unlikely to both be at their maximum or minimum value at the same time.