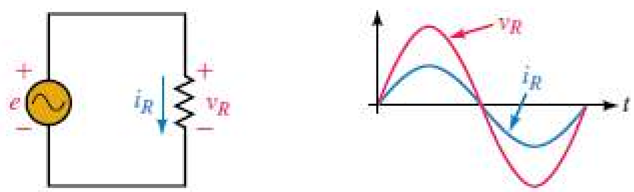
Chapter six: The Basic Elements and Phasors

6.1 R, L, and *C* Circuits with Sinusoidal Excitation

R, *L*, and *C* circuit elements each have quite different electrical properties. Resistance, for example, opposes current, while inductance opposes changes in current, and capacitance opposes changes in voltage.

1. **Resistance and Sinusoidal AC**
   For a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm's law, as shown in figure 6.1:



(a) Source voltage is a sine wave. Therefore, v_R is a sine wave.

(b) $i_R = v_R/R$. Therefore i_R is a sine wave.

Figure 6.1: Ohm s law applies to resistors

$$i_R = \frac{v_R}{R} = \frac{V_m \sin wt}{R} = \frac{V_m}{R}\sin wt = I_m \sin wt$$

Where

$$I_m = \frac{V_m}{R}$$

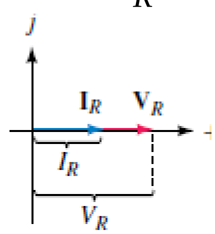
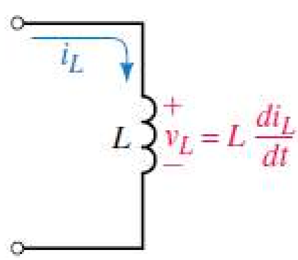


Figure 6.2: For resistor, voltage and current in phase

2. **Inductance and sinusoidal AC**
   For inductance



$$v_L = L\frac{di_L}{dt}$$

Since $i_L = I_m \sin wt$

$$\frac{di_L}{dt} = I_m(w\cos wt)$$

$$v_L = L\frac{di_L}{dt} = LwI_m \cos wt$$

Or

$$v_L = V_m \sin(wt + 90^\circ)$$

For an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°, as show in figure 6.3 (a) and (b)

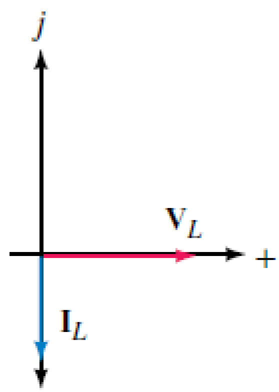


Figure 6.3 (a): Current I_L always lags voltage by 90°

Where

$$V_m = wLI_m$$

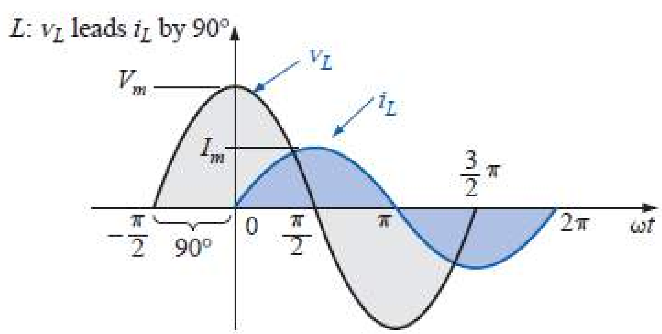


Figure 6.3 (b): for a pure inductor, the voltage across the coil leads the current through the coil by 90°

Inductive reactance

$$\frac{V_m}{I_m} = wL$$

$$\frac{V_m}{I_m} = X_L$$

The quantity wL, called the reactance of an inductor, is symbolically represented by X_L and is measured in ohms:

$$\boxed{X_L = wL = 2\pi fL} \quad \text{(ohms, } \Omega\text{)}$$

Where w in radians per second and L in henries.

3. **Capacitance and sinusoidal AC**
   For capacitance, current proportional to the rate of change of voltage, i.e.,

$$i_c = C\frac{dv_c}{dt} = C\frac{d}{dt}(V_m \sin wt) = wCV_m \cos wt = I_m \cos wt$$

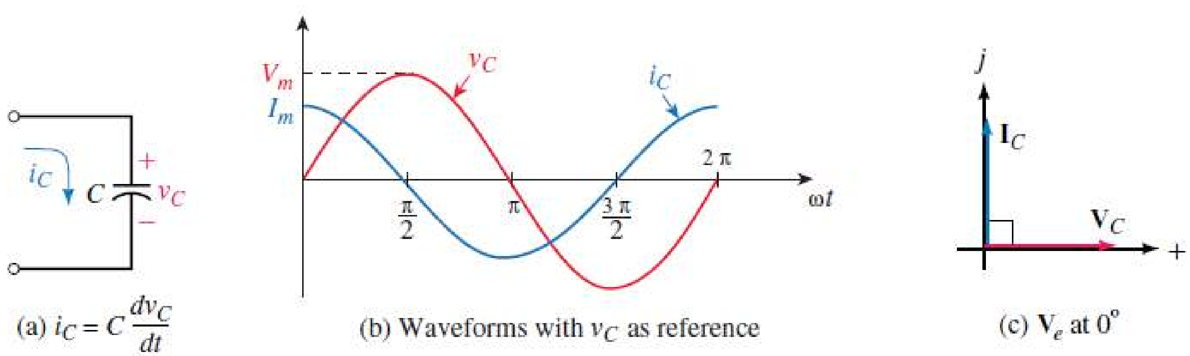


Figure 6.1: for capacitance, current always leads voltage by 90°

Capacitive reactance

From the last equation, consider the relationship between maximum capacitor voltage and current magnitudes

$$I_m = wCV_m \rightarrow \frac{V_m}{I_m} = \frac{1}{wc}$$

The ratio of V_m to I_m is defined as capacitive reactance and is given by the symbol X_c. That is

$$X_c = \frac{V_m}{I_m} \quad (\Omega)$$

Since

$$\frac{V_m}{I_m} = \frac{1}{wc},$$

we also get

$$\boxed{X_c = \frac{1}{wC}} \quad (\Omega)$$

Where w in radian per second and C in farads.

$$I_m = \frac{V_m}{X_c}$$

And

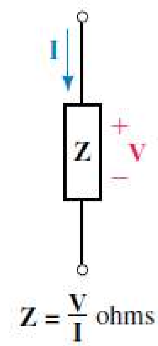
$$V_m = I_m X_c$$

6.2 Impedance Concept

Impedance

The opposition that circuit elements present to current in the phasor domain is defined as its ***impedance***.

For example, the impedance of the element of figure 6.4 is the ratio of its voltage phasors to its current phasor.



The impedance is symbolically represented by Z . Thus,

$$Z = \frac{V}{I} \quad \text{(ohms)}$$

Since phasor voltages and currents are complex, Z is also complex. That is,

$$Z = \frac{V}{I}\angle\theta$$

Where V and I are rms magnitudes of **V** and **I respectively, and** θ is the angle between them

$$Z = Z\angle\theta$$

Once the impedance of a circuit is known , the current and voltage can be determined using:

$$I = \frac{V}{Z}$$

And

$$V = IZ$$

Let us now determine impedance for the basic circuit elements $R, L,$ and C .

Resistance

For pure resistance as shown in figure 6.5:

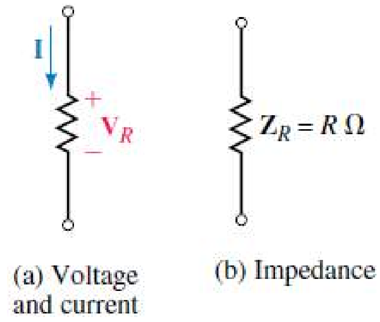


Figure 6.5: Impedance of a pure resistance

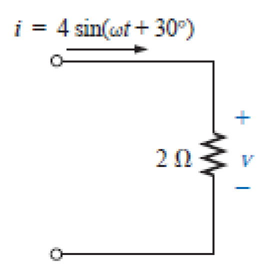
$$Z_R = \frac{V_R\angle\theta}{I\angle\theta} = \frac{V_R}{I}\angle 0^\circ = R\angle 0^\circ$$

Thus the impedance of a resistor is just its resistance. This is

$$\boxed{Z_R = R\angle 0^\circ}$$

Example 6.1:

Find the voltage *v* for the circuit shown below:



Solution:

$i = 4\sin(wt + 30^\circ) \rightarrow$ phasor form $I = 4 \times 0.707\angle 30^\circ = 2.828\angle 30^\circ$ A

$V = IZ_R = (I\angle\theta)(R\angle 0^\circ) = (2.828\angle 30^\circ)(2\angle 0^\circ) = 5.656\angle 30^\circ$ V

And

$v = \sqrt{2} \times (5.656)\sin(wt + 30^\circ)$ V

Inductance

For pure inductance (see figure 6.6), current lags voltage by 90°, therefore:

$$Z_L = \frac{V_L \angle\theta}{I\angle\theta} = \frac{V_R \angle 0^\circ}{I\angle -90^\circ} = wL\angle 90^\circ = jwL$$

$$\boxed{Z_L = jwL = jX_L = X_L\angle 90^\circ}$$

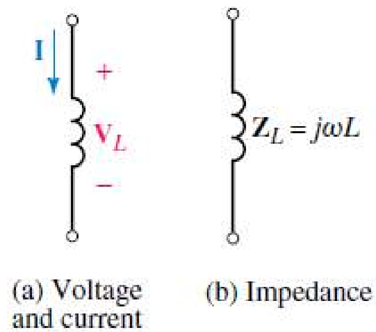
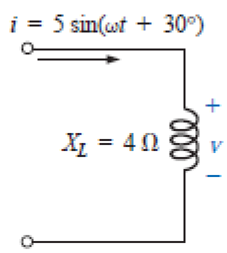


Figure 6.6: Impedance of a pure inductance

Example 6.2:

Find the voltage v for the circuit shown below:



Solution:

$i = 5\sin(wt + 30^\circ) \rightarrow I = 0.707\times 5\angle 30^\circ = 3.535\angle 30^\circ$ A

$V = IZ_L = (I\angle\theta)(X_L\angle 90^\circ) = (3.535\angle 30^\circ)(4\angle 90^\circ) = 14.14\angle 120^\circ$ V

And

$v = \sqrt{2}(14.14)\sin(wt + 120^\circ)$ V

Capacitance

For a pure capacitance (figure 6.7), current leads voltage by 90°

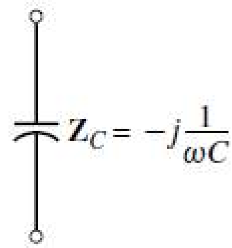


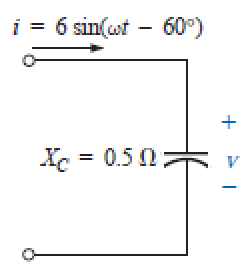
Figure 6.7: Impedance of a pure capacitance

$$Z_C = \frac{V_C\angle\theta}{I\angle\theta} = \frac{V_R\angle 0^{\circ}}{I\angle 90^{\circ}} = \frac{1}{wC}\angle -90^{\circ} = -j\frac{1}{wC}$$

$$\boxed{Z_C = -j\frac{1}{wC} = -jX_C} \quad \text{(ohms)}$$

Example 6.3:

Find the voltage v for the circuit shown below:



Solution:

$i = 6\sin(wt - 60^{\circ}) \rightarrow$ phasor form $I = 0.707\times 6\angle -60^{\circ} = 4.242\angle -60^{\circ}$ A

$V = IZ_L = (I\angle\theta)(X_C\angle -90^{\circ}) = (4.242\angle -60^{\circ})(0.5\angle -90^{\circ}) = 2.121\angle -150^{\circ}$ V

And

$v = \sqrt{2}(2.121)\sin(wt - 150^{\circ}) = 3\sin(wt - 150^{\circ})$ A

6.3 AVERAGE POWER AND POWER FACTOR

If we take the general case and use the following for *v* and *i*:

$$v(t) = V_m \sin(wt + \theta_v^\circ)$$

$$i(t) = I_m \sin(wt + \theta_i^\circ)$$

Since the power is defined by

$$p = vi$$

$$p = V_m \sin(wt + \theta_v^\circ) I_m \sin(wt + \theta_i^\circ)$$

Since

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$p = \frac{V_m I_m}{2}\left[\cos(\theta_v^\circ - \theta_i^\circ) - \cos(2wt + \theta_v^\circ + \theta_i^\circ)\right]$$

$$p = \frac{V_m I_m}{2}\left[\cos(\theta_v^\circ - \theta_i^\circ)\right] - \frac{V_m I_m}{2}\left[\cos(2wt + \theta_v^\circ + \theta_i^\circ)\right]$$

The average value of $\frac{V_m I_m}{2}\left[\cos(2wt + \theta_v^\circ + \theta_i^\circ)\right]$ is zero over one cycle, thus, The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power,**

$$p = \frac{V_m I_m}{2}\left[\cos(\theta_v^\circ - \theta_i^\circ)\right]$$

The angle $(\theta_v^\circ - \theta_i^\circ)$ is the phase angle between *v* and *i*. Since $\cos(-\alpha) = \cos\alpha$, thus

$$\boxed{p = \frac{V_m I_m}{2}\cos\theta} \quad \text{(watts, W)}$$

Where p is the average power in watts. The last equation can be written

$$p = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

Since $V_{eff} = \frac{V_m}{\sqrt{2}}$, and $I_{eff} = \frac{I_m}{\sqrt{2}}$

$$\boxed{p = V_{eff} I_{eff} \cos\theta}$$

Power Factor

In the equation $P = \frac{V_m I_m}{2} \cos\theta$, the factor that has significant control over the delivered power level is the $\cos\theta$. No matter how large the voltage or current, if $\cos\theta = 0$, the power is zero; if $\cos\theta = 1$, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\text{Power Factor } = F_P = \cos\theta$$

Or

$$F_P = \cos\theta = \frac{P}{V_{eff} I_{eff}}$$

The terms *leading* and *lagging* are often written in conjunction with the power factor. *They are defined by the current through the load.* If the current leads the voltage across a load, the load has a **leading power factor.** If the current lags the voltage across the load, the load has a **lagging power factor.**

Example 6.4:

Determine the average power delivered to networks having the following input voltage and current:

1) $v(t) = 100\sin(wt + 40^\circ)$, $i(t) = 20\sin(wt + 70^\circ)$

2) $v(t) = 150\sin(wt - 70^\circ)$, $i(t) = 3\sin(wt - 50^\circ)$

Solution:

1) $P = \frac{V_m I_m}{2} \cos\theta$

$\theta = 40^\circ - 70^\circ = -30^\circ$

$P = \frac{100 \times 20}{2} \cos(-30^\circ) = 866W$

2) $P = \frac{V_m I_m}{2} \cos\theta$

$\theta = -70^\circ - (-50^\circ) = -20^\circ$

$P = \frac{150 \times 3}{2} \cos(-20^\circ) = 211.43W$

Chapter seven: Series and Parallel ac Circuits

7.1 SERIES ac CIRCUITS

The total impedance of a system is the sum of the individual impedances:

$$Z_T = Z_1 + Z_2 + Z_3 + ... + Z_N$$

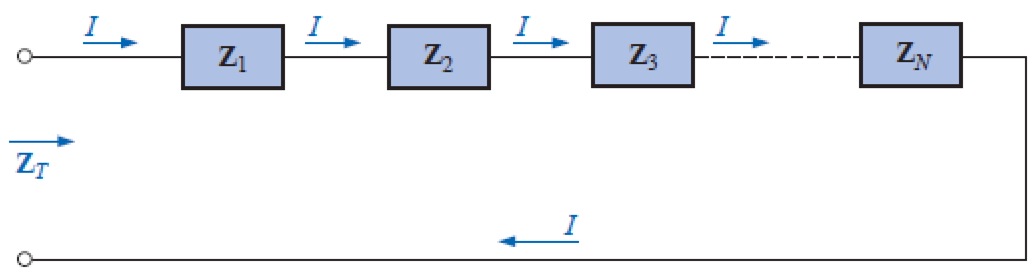
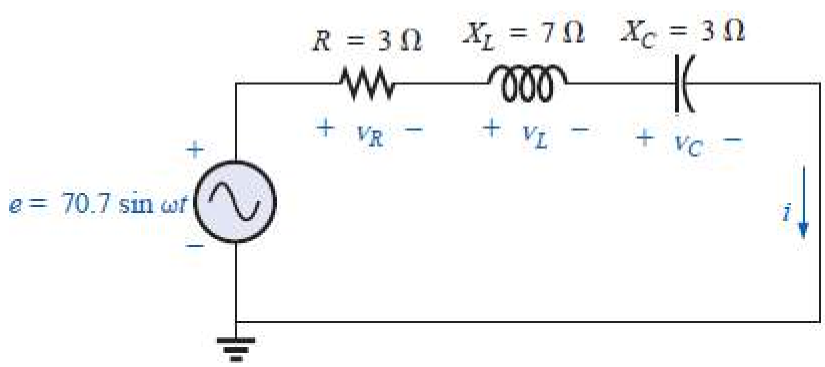


Figure 7.1: Series impedances

Example 7.1:

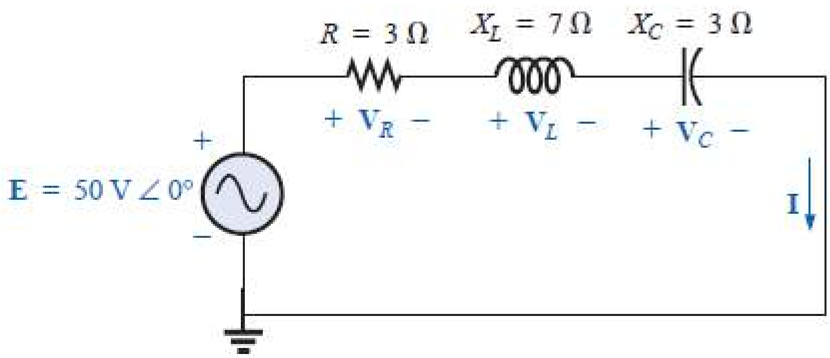
For the circuit shown below, find

1. The unknown voltages and current values.
2. The total power
3. Power factor



Solution:

Convert the voltage source s value to phasor form, as shown below:



1)

$Z_T = Z_1 + Z_2 + Z_3 = R\angle 0^\circ + X_L\angle 90^\circ + X_C\angle -90^\circ$

$$Z_T = 3 + j7 - j3 = 3 + j4 \ \Omega = 5 \ \Omega\angle 53.13^\circ$$

$I = \dfrac{E}{Z_T} = \dfrac{50\angle 0^\circ}{5\angle 53.13^\circ} = 10 \ A\angle -53.13^\circ$

$V_R = IZ_R = 10\angle -53.13^\circ \times 3\angle 0^\circ = 30 \ V\angle -53.13^\circ$

$V_L = IZ_L = 10\angle -53.13^\circ \times 7\angle 90^\circ = 70 \ V\angle 36.87^\circ$

$V_C = IZ_C = 10\angle -53.13^\circ \times 3\angle -90^\circ = 30 \ V\angle -143.13^\circ$

2) $P_T = E_{eff} I_{eff} \cos\theta = EI\cos\theta = (5)(10)\cos(53.13) = 300 \ W$

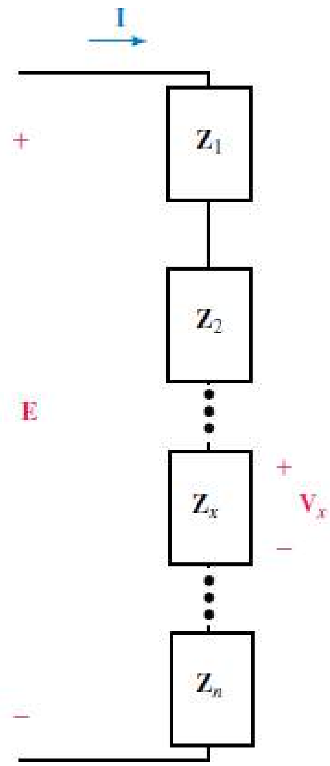
Or $P_T = I^2 R = (10)^2 3 = 300 \ W$

3) Power factor $= \cos\theta = \cos(53.13^\circ) = 0.6$ lagging

7.1.1 Kirchhoff's Voltage Law and the Voltage Divider Rule

Voltage divider rule

$$V_x = \frac{EZ_x}{Z_T}$$

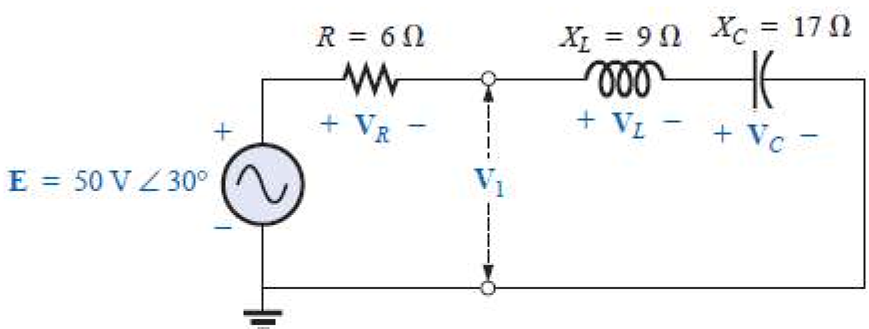


Kirchhoff's voltage law for ac circuits may be stated as follows:

The phasor sum of voltage drops and voltage rises around a closed loop is equal to zero.

Example 7.2:

Using voltage divider rule, find the unknown voltages V_R, V_L, V_C and V_1



Solution:

$$V_R = \frac{EZ_R}{Z_T} = \frac{EZ_R}{Z_R + Z_L + Z_C} = \frac{(50\angle 30^\circ)(6\angle 0^\circ)}{6\angle 0^\circ + 9\angle 90^\circ + 17\angle -90^\circ} = \frac{300\angle 30^\circ}{6 + j9 - j17} = \frac{300\angle 30^\circ}{10\angle -53.13^\circ} = 30\ V\angle 83.13^\circ$$

$$V_L = \frac{EZ_L}{Z_T} = \frac{(50\angle 30^\circ)(9\angle 90^\circ)}{10\angle -53.13^\circ} = \frac{450\angle 120^\circ}{10\angle -53.13^\circ} = 45\ V\angle 173.13^\circ$$

$$V_C = \frac{EZ_C}{Z_T} = \frac{(50\angle 30^\circ)(17\angle -90^\circ)}{10\angle -53.13^\circ} = \frac{850\angle -60^\circ}{10\angle -53.13^\circ} = 85\ V\angle -6.87^\circ$$

$$V_1 = \frac{E(Z_C + Z_L)}{Z_T} = \frac{(50\angle 30^\circ)(-j17 + j9)}{10\angle -53.13^\circ} = \frac{(50\angle 30^\circ)(8\angle -90^\circ)}{10\angle -53.13^\circ} = \frac{400\angle -60^\circ}{10\angle -53.13^\circ} = 40\ V\angle -6.87^\circ$$

We can also find V_1 using Kirchhoff's voltage law:

$$E - V_R - V_1 = 0 \rightarrow$$

$$V_1 = E - V_R = 50\angle 30^\circ - 30\angle 83.13^\circ = 43.301 + j25 - (3.588 + j29.784) = 39.713 - j4.784 = 40\ V\angle -6.87^\circ$$

7.2 Parallel ac circuits

Admittance

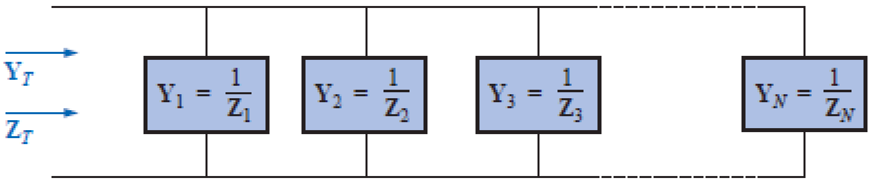
In ac circuits, we define **admittance** (**Y**) as being equal to 1/**Z**. The unit of measure for admittance as defined by the SI system is *siemens*, which has the symbol S. Admittance is a measure of how well an ac circuit will *admit*, or allow, current to flow in the circuit.

The total admittance of a circuit can also be found by finding the sum of the parallel admittances.

$$Y_T = Y_1 + Y_2 + Y_3 + \dots\dots\dots + Y_N$$

The total impedance Z_T of the circuit is then $\frac{1}{Y_T}$; that is, for the network shown below:

$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots\dots\dots + \frac{1}{Z_N}$$



For two impedances in parallel,

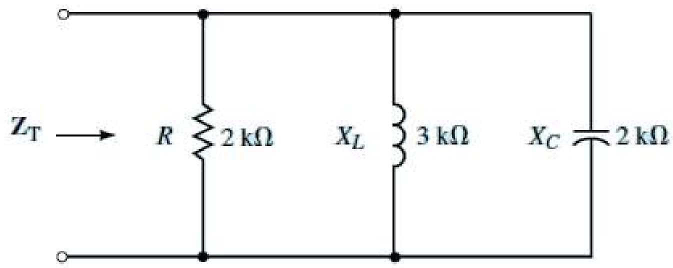
$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

For three parallel impedances,

$$Z_T = \frac{Z_1 Z_2 Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

Example 7.3:

For the circuit shown below, find the equivalent impedance



Solution:

$$Z_T = \frac{Z_1 Z_2 Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} = \frac{(2k\Omega\angle 0^\circ)(3k\Omega\angle 90^\circ)(2k\Omega\angle -90^\circ)}{(2k\Omega\angle 0^\circ)(3k\Omega\angle 90^\circ) + (2k\Omega\angle 0^\circ)(2k\Omega\angle -90^\circ) + (3k\Omega\angle 90^\circ)(2k\Omega\angle -90^\circ)}$$

$$= \frac{12\times 10^9\angle 0^\circ}{6\times 10^6\angle 90^\circ + 4\times 10^6\angle -90^\circ) + 6\times 10^6\angle -90^\circ)} = \frac{12\times 10^9\angle 0^\circ}{6\times 10^6 + j2\times 10^6}$$

$$= \frac{12\times 10^9\angle 0^\circ}{6\times 10^6\angle 90^\circ + 4\times 10^6\angle -90^\circ) + 6\times 10^6\angle -90^\circ)} = \frac{12\times 10^9\Omega\angle 0^\circ}{6.325\times 10^6\angle 18.43^\circ} = 1.90k\Omega\angle -18.43^\circ$$

7.3.1 Kirchhoff's Current Law and the Current Divider Rule

KCL may be stated as follows:

The summation of current phasors entering and leaving a node is equal to zero.

The current divider rule for ac circuits has the same form as for dc circuits with the notable exception that currents are expressed as phasors. For a parallel network as shown in Figure 7–1, the current in any branch of the network may be determined using either admittance or impedance.

$$I_x = \frac{Y_1}{Y_T} I \qquad \text{or} \qquad I_x = \frac{Z_T}{Z_x} I$$

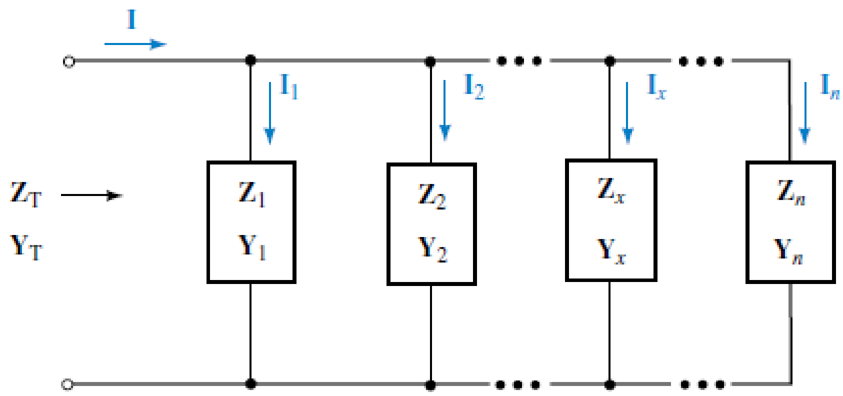


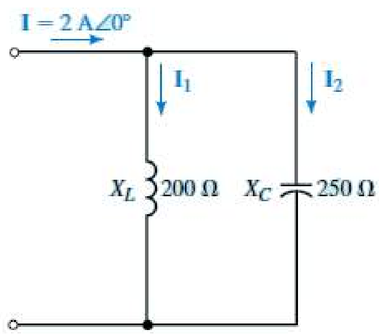
Figure 7-1:

For two branches in parallel the current in either branch is determined from the impedances as

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I$$

Example 7.4:

For the circuit below, find I_1 and I_2



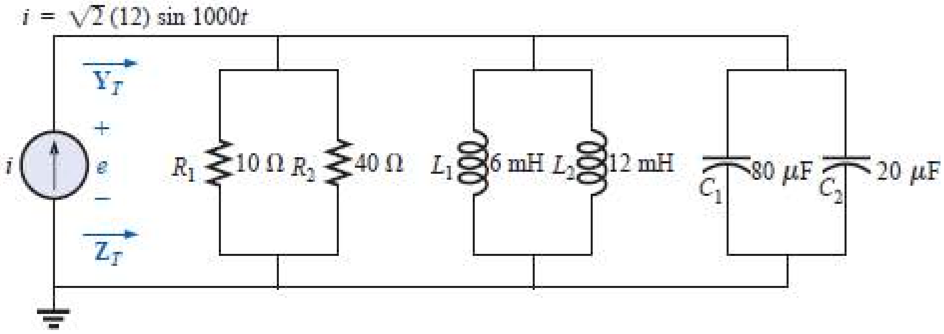
Solution:

$$I_1 = (\frac{250\ \Omega\angle -90^\circ}{j200 - j250})(2\ A\angle 0^\circ) = (\frac{250\ \Omega\angle -90^\circ}{50\angle -90^\circ})(2\ A\angle 0^\circ) = 10\ A\angle 0^\circ$$

$$I_2 = (\frac{200\ \Omega\angle 90^\circ}{j200 - j250})(2\ A\angle 0^\circ) = (\frac{250\ \Omega\angle 90^\circ}{50\angle -90^\circ})(2\ A\angle 0^\circ) = 8\ A\angle 180^\circ$$

Example 7.5:

For the circuit shown below, Determine Y_T



Solution

$$R_T = 10\Omega // 40\Omega = 8\Omega$$

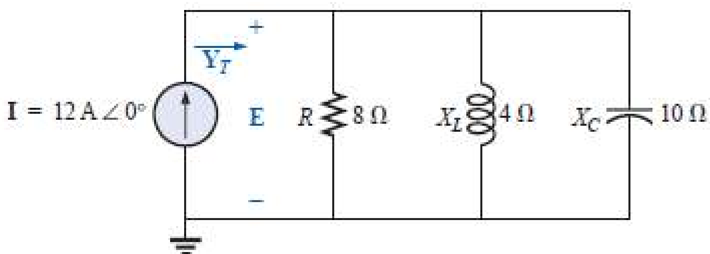
$$L_T = 6mH // 12mH = \frac{6mH \times 12mH}{6mH + 12mH} = 4mH$$

$$C_T = 80\mu F + 20\mu F = 100\mu F$$

$$X_L = wL = (1000 rad/\sec)(4mH) = 4\ \Omega$$

$$X_c = \frac{1}{wC} = \frac{1}{(1000 rad/\sec)(100\mu F)} = 10\ \Omega$$

The network is redrawn in the figure below with phasor notation.



The total admittance is

$$Y_T = Y_R + Y_L + Y_C$$

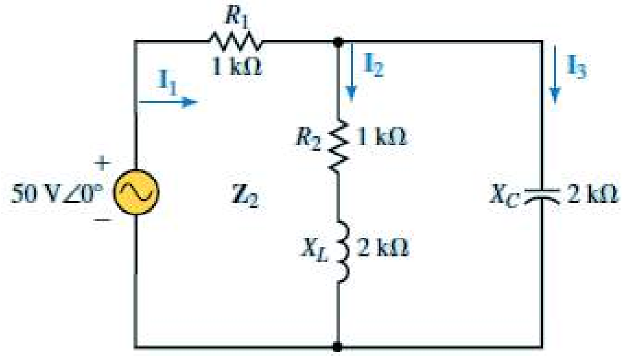
$$= \frac{1}{R\angle 0^\circ} + \frac{1}{X_L\angle 90^\circ} + \frac{1}{X_c\angle -90^\circ} = \frac{1}{8}\angle 0^\circ + \frac{1}{4}\angle -90^\circ + \frac{1}{10}\angle 90^\circ$$

$$= 0.125\ S\angle 0^\circ - j0.25\ S + j0.1\ S$$

$$= 0.125\ S - j0.15\ S = 0.195\ S\angle -50.194^\circ$$

7.4 Series-Parallel Circuits

Example 7.6:
For the circuit shown below



a) Find Z_T
b) Determine the currents I_1, I_2 and I_3
c) Calculate the total power provided by the voltage source.
d) Determine the average powers P_1, P_2 and P_3 dissipated by each of the impedances.

Solution:

a) $Z_T = Z_1 + (Z_2 // Z_3)$

$$Z_2 // Z_3 = \frac{(1k\Omega + j2k\Omega)(-j2k\Omega)}{1k\Omega + j2k\Omega - j2k\Omega} = \frac{(2.236k\Omega\angle 63.43^\circ)(2k\Omega\angle -90^\circ)}{1k\Omega\angle 0^\circ} = 4.472k\Omega\angle -26.57^\circ = 4k\Omega - j2k\Omega$$

$$Z_T = 5k\Omega - j2k\Omega = 5.385k\Omega\angle -21.80^\circ$$

b) $$I_1 = \frac{V}{Z_T} = \frac{50V\angle 0^\circ}{5.385k\Omega\angle -21.80^\circ} = 9.285\ mA\angle 21.80^\circ$$

$$I_2 = \frac{I_1 Z_3}{Z_2 + Z_3} = \frac{(9.285\ mA\angle 21.80^\circ)(2k\Omega\angle -90^\circ)}{1k\Omega + j2k\Omega - j2k\Omega} = 18.57\ mA\angle -68.20^\circ$$

$$I_3 = \frac{I_1 Z_2}{Z_2 + Z_3} = \frac{(9.285\ mA\angle 21.80^\circ)(1k\Omega + j2k\Omega)}{1k\Omega + j2k\Omega - j2k\Omega} = 20.761\ mA\angle 85.23^\circ$$

c) $P = VI\cos\theta = (50\ V)(9.285\ mA)\cos(21.80^\circ) = 431\ mW$

d) Because only the resistors will dissipate power, we may use $P = I^2 R$

$$P_1 = (9.285\ mA)^2(1\ k\Omega) = 86.2\ mW$$

$$P_2 = (18.57\ mA)^2(1\ k\Omega) = 344.8\ mW$$

$$P_3 = 0$$