# Chapter Five: AC Fundamentals

## 5.1 Introduction

**Alternating currents (ac)** are currents that alternate in direction (usually many times per second). Such currents are produced by voltage sources whose polarities alternate between positive and negative. By convention, alternating currents are called ***ac currents*** and alternating voltages are called ***ac voltages***.

Figure 5.1 shows a voltage waveform. Starting at zero, the voltage increases to a positive maximum, decreases to zero, changes polarity, increases to a negative maximum, then returns again to zero. One complete variation is referred to as a **cycle.** Since the waveform repeats itself at regular intervals as in (b), it is called a **periodic** waveform.

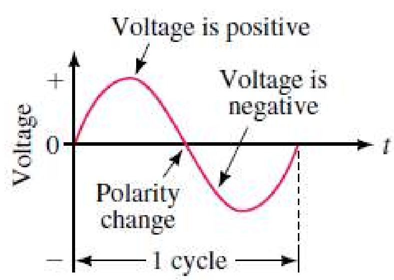



Figure 5.1: Variation of voltage versus time

## 5.2 Definitions

***Waveform*****:** It is a graph showing the manner in which an alternating quantity changes with time.

***Frequency:***
The number of cycles per second of a waveform is defined as its **frequency.**

| 1 Hz =1 cycle per second |
|---|

The examples depicted in figure 5.3 represent 1 Hz, 2 Hz, and 60 Hz respectively.

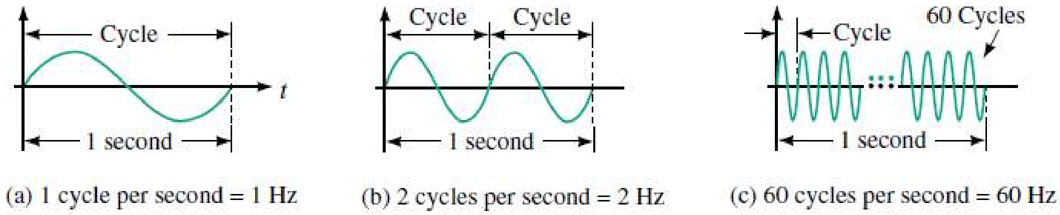



Figure 5–2: Frequency is measured in hertz (Hz).

### *Period*

The **period (or the time period),** *T*, of a waveform, (Figure 5–3) is the duration of one cycle. It is the inverse of frequency.

$$T=\frac{1}{f} \qquad \text{(Second)}$$

And

$$f=\frac{1}{T} \qquad \text{(Hz)}$$

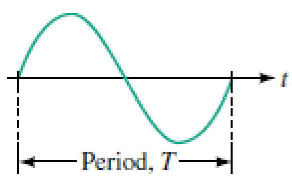
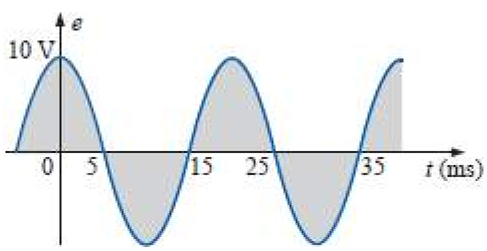



Figure 5.3: Period T is the duration of one cycle, measured in second

***Example 5.1:***

Determine the frequency of the waveform of the figure shown below




**Solution:**

From the figure, Time period (T) =20 ms

$$f=\frac{1}{T}=\frac{1}{20\times10^{-3}}=50Hz$$

## Amplitude and Peak-to-Peak Value

The **amplitude** of a sine wave is the distance from its average to its peak. Thus, the amplitude of the voltage in Figures 5–4 is $E_m$

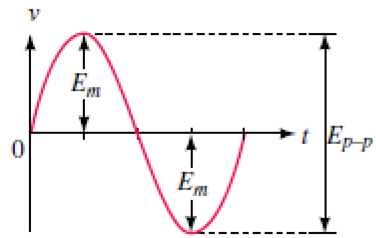



Figure 5-4: Alternating waveform

***Peak-to-peak voltage*** is also indicated in Figure 5–4. Denoted by *Ep-p* or *Vp-p,* the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.

***Peak Value*****:** The **peak value** of a voltage or current is its maximum value with respect to zero.

**Instantaneous values** are the values of the alternating quantities at any instant of time. They are represented by small letters, i, υ, e, etc.

**Angular Velocity ($w$):** The rate at which the generator coil rotates is called its **angular velocity.** If the coil rotates through an angle of 30° in one second, for example, its angular velocity is 30° per second.

In practice, $w$ is usually expressed in radians per second, where radians and degrees are related by the identity:

$$2\pi \text{ radians} = 360^\circ$$

To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ}$, while to convert from radians to degrees, multiply by $\frac{180^\circ}{\pi}$.

***Example 5.2:***

a. Convert 315° to radians.

b. Convert $\frac{5\pi}{4}$ radians to degrees.

Solution:

a. $\alpha_{radian} = \frac{180^\circ}{\pi} \times 315^\circ = 5.5$ rad.

b. $\alpha_{\deg ree} = \frac{\pi}{180^\circ} \times \frac{5\pi}{4} = 255^\circ$

## 5.3 Sinusoidal Voltages and Currents as Functions of Time:

The basic mathematical formats for the sinusoidal voltage and currents are:

$$v(t) = V_m \sin wt$$

$$i(t) = I_m \sin wt$$

Where:

$V_m$ : The maximum value of the voltage

$I_m$ : The maximum value of the current

$w$ : The angular frequency

$$\boxed{w = 2\pi f = \frac{2\pi}{T}}$$

***Example 5.3:***

A 100-Hz sinusoidal voltage source has amplitude of 150 volts. Write the equation for *e* as a function of time.

**Solution:**

$v(t) = V_m \sin wt$

$V_m = 150V$

$w = 2\pi f = 2\pi \times 100 = 200\pi$

Thus:

$v(t) = 150 \sin 200\pi t$ V

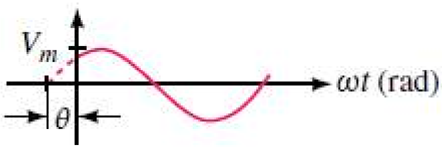
## 5.4. Voltages and Currents with Phase Shifts

If a sine wave does not pass through zero at $t$ = 0 s as in Figure 5–6, it has a **phase shift.** Waveforms may be shifted to the left or to the right (see Figure 5–6). For a waveform shifted left as in (a),

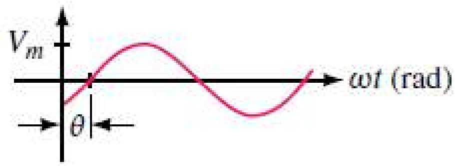
$$v(t) = V_m \sin(wt + \theta)$$

While, for a waveform shifted right as in (b),

$$v(t) = V_m \sin(wt - \theta)$$




(a) $v(t) = V_m \sin(wt + \theta)$

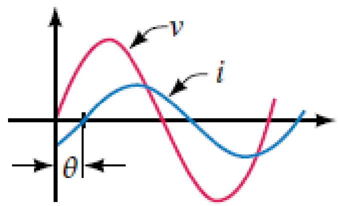



(b) $v(t) = V_m \sin(wt - \theta)$

Figure 5.6: Waveforms with phase shifts

### *5.4.1 Phase difference*

When two alternating quantities with the same frequency have different zero points, they are said to have phase difference.




The angle between the two zero points is the angle of phase difference $\phi$ and it is measured in radians or degrees. The voltage waveform is called leading and the current waveform is called lagging.

### 5.4.2 Phase relations

The terms *lead* and *lag* are used to indicate the relationship between two sinusoidal waveforms of the *same frequency* plotted on the same set of axes. In Fig. 5.7, the cosine curve is said to *lead* the sine curve by 90°, and the sine curve is said to *lag* the cosine curve by 90°. The 90° is referred to as the phase angle between the two waveforms.

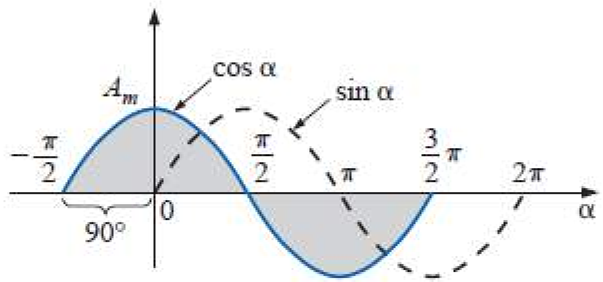



Figure 5.6: Phase relationship between a sine wave and a cosine wave

If both waveforms cross the axis at the same point with the same slope, they are *in phase*
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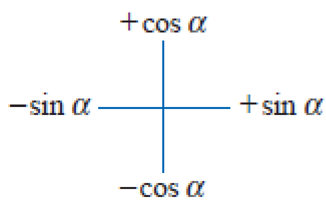



Figure 5.7: *Graphic tool for finding the relationship between specific sine and cosine functions.*

The geometric relationship between various forms of the sine and cosine functions can be derived from Fig. 5.7. For instance, starting at the sin a position, we find that $\cos\theta$ is an additional 90° in the counterclockwise direction. Therefore, $\cos\theta = \sin(\theta + 90^\circ)$. For $-\sin\theta$ we must travel 180° in the counterclockwise (or clockwise) direction so that $-\sin\theta = \sin(\theta - 180^\circ)$, and so on, as listed below:

$$\cos\theta = \sin(\theta + 90^\circ)$$
$$\sin\theta = \cos(\theta - 90^\circ)$$
$$-\sin\theta = \sin(\theta \pm 180^\circ)$$
$$-\cos\theta = \sin(\theta + 270^\circ) = \sin(\theta - 90^\circ)$$

In addition, one should be aware that

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

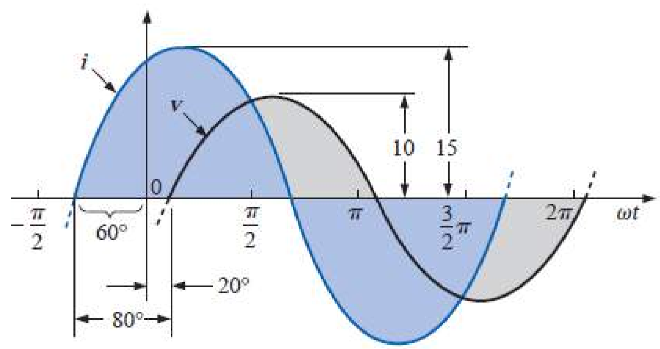
***Example 5.4:***
What is the phase relationship between the sinusoidal waveforms of each of the following sets?

1) $i = 15\sin(wt + 60^\circ)$, $v = 10\sin(wt - 20^\circ)$
2) $i = 2\cos(wt + 10^\circ)$ , $v = 3\sin(wt - 10^\circ)$

**Solution:**

1) $i = 15\sin(wt + 60^\circ)$, $v = 10\sin(wt - 20^\circ)$

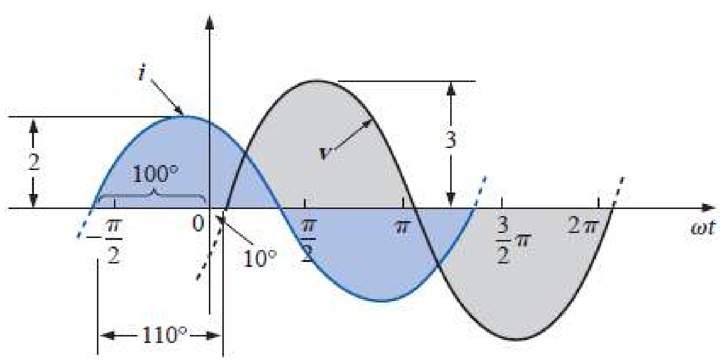
   **i leads v by 80°, or v lags i by 80°.**



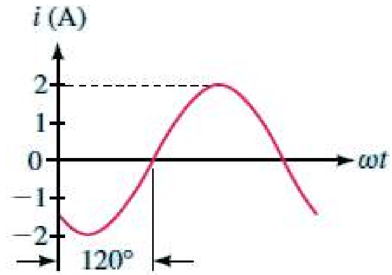

2) $i = 2\cos(wt + 10^\circ) = 2\sin(wt + 10^\circ + 90^\circ) = 2\sin(wt + 100^\circ)$

   $v = 3\sin(wt - 10^\circ)$

   ***i* leads *v* by 110°, or *v* lags *i* by 110°.**



***Example 5.5:***
Determine the equation for the waveform shown below, given *f* =60 Hz.




Solution:

$I_m = 2$ A and $w = 2\pi f = 2\pi(60) = 377$ rad/sec.

$i = I_m \sin(wt - \theta) = 2\sin(377t - 120^\circ)$ A

***Example 5.6:***
The current in an a.c. circuit at any time *t* seconds is given by:
$i$ =120sin (100$\pi t$ +0.36) amperes. Find:
(a) The peak value, the periodic time, the frequency and phase angle relative to 120 sin100$\pi t$
(b) The value of the current when $t$ =0
(c) The value of the current when $t$ =8ms
(d) The time when the current first reaches 60A

**Solution**:

Peak value = 120A

Periodic time, $T = \frac{2\pi}{w} = \frac{2\pi}{100\pi} = \frac{1}{50} = 0.02$ s

Frequency, $f = \frac{1}{T} = \frac{1}{0.02} = 50$ Hz

Phase angle $= 0.36 rad. = 0.36 \times \left(\frac{180}{\pi}\right) = 20.63^\circ$ leading

(b) When $t$ =0, $i$=120sin (0+0.36) =120sin20.63° = 49.3A

(c) When $t$ =8ms, $i = 120\sin\left[100\pi\left(8\times10^{-3}\right) + 0.36\right] = 120\sin\left(2.8733\right) = 31.8$ A

(d) When $i$ =60A, 60=120sin (100$\pi t$ +0.36)

$$\sin\left[100\pi t + 0.36\right] = \frac{60}{120}$$

$$\left[100\pi t + 0.36\right] = \sin^{-1}\left(0.5\right) = 30^\circ = \frac{\pi}{6} rad = 0.5236 rads$$

$$t = \frac{0.5236 - 0.36}{100\pi} = 0.521$$

## 5. 5. Average value

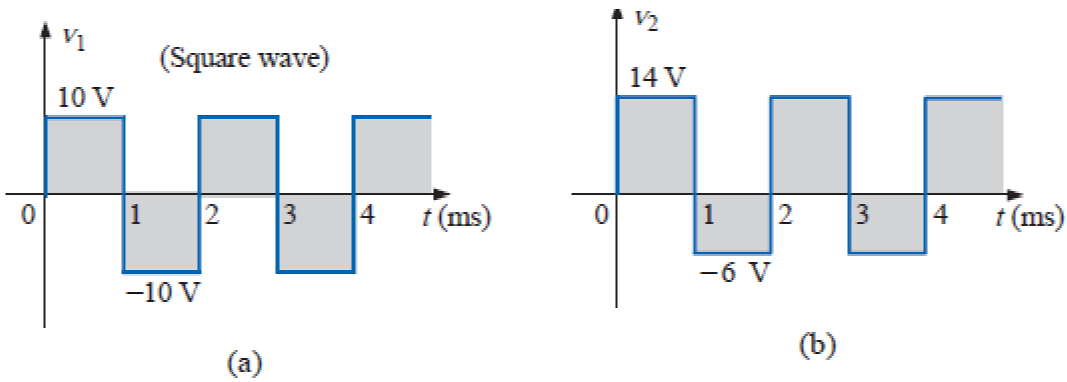
To find the average value of a waveform, divide the area under the waveform by the length of its base. Average values are also called **dc values,** because dc meters indicate average values rather than instantaneous values.

In general, average value is given by:

$$verage\ value = \frac{\text{area under curve}}{\text{length of base}}$$

***Example 5.7:***

Determine the average value of the waveforms of the figures shown below:




Solution:

a) $verage\ value = \frac{\text{area under curve}}{\text{length of base}}$

$$\text{Average value}=\frac{(10)(1ms)\text{-}(10)(1ms)}{2ms}=0\,\text{V}$$

b) $verage\ value = \frac{\text{area under curve}}{\text{length of base}}$

$$\text{Average value}=\frac{(14)(1ms)\text{-}(6)(1ms)}{2ms}=4\ \text{V}$$

### Sine Wave Averages

Because a sine wave is symmetrical, its area below the horizontal axis is the same as its area above the axis; thus, over a full cycle its net area is zero, independent of frequency and phase angle. The average of half a sine wave, however, is not zero. Consider Figure 5–8. The area under the half-cycle may be found using calculus as:

$$\text{area}=\int_0^T f(t)dt=\int_0^\pi A_m \sin t=\left[-A_m \cos t\right]_0^\pi=2A_m$$

Where:

$A_m$ : is the maximum value

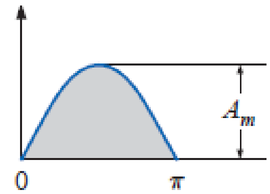


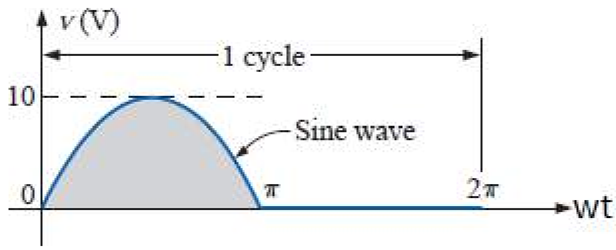

Figure 5.8: Area under a half-cycle

Therefore,

$$\text{Average value} = \frac{2A_m}{\pi} = 0.637A_m$$

**Example 5.8:**

Determine the average value of the waveform shown below:




**Solution:**

$$verage\ value = \frac{\text{area under curve}}{\text{length of base}}$$

$$= \frac{2A_m + 0}{2\pi} = \frac{2 \times 10}{2\pi} \cong 3.18\,\text{V}$$

## 5.5 Effective value (or root mean square)

An effective value is an equivalent dc value: it tells you how many volts or amps of dc that a time-varying waveform is equal to in terms of its ability to produce average power. Effective values depend on the waveform.

### Effective Values for Sine Waves

The effective value of a waveform can be determined using the circuits of Figure 5–9. Consider a sinusoidally varying current, $i(t)$. By definition, the effective value of $i$ is that value of dc current that produces the same average power. Consider (b). Let the dc source be adjusted until its average power is the same as the average power in (a). The resulting dc current is then the effective value of the current of (a). To determine this value, determine the average power for both cases, and then equate them.

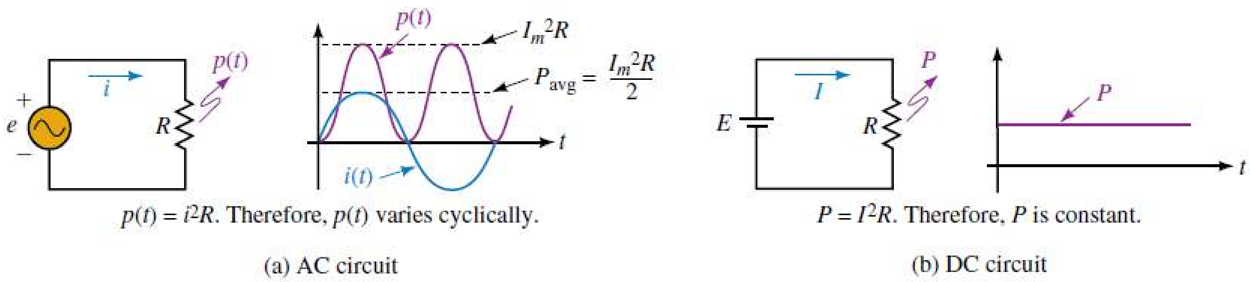


Figure 5.9: Determining the effective value of sinusoidal ac.

First, consider the dc case. Since current is constant, power is constant, and average power is:

$$P_{avg} = P = I^2R\text{.................}(1)$$

Now, consider the ac circuit:

$$p(t) = i^2R$$

$$= (I_m \sin wt)^2 R = I_m^2 R \sin^2 wt$$

But, $\sin^2 wt = \frac{1}{2}(1 - \cos 2wt)$

$$= \frac{I_m^2}{R}\left[\frac{1}{2}(1 - \cos 2wt)\right]$$

$$= \frac{I_m^2 R}{2} - \frac{I_m^2 R}{2}\cos 2wt$$

Since the average power of $\cos 2wt$ is zero, therefore:

$$P_{avg} = \text{Average of } p(t) = \frac{I_m^2 R}{2}\text{..................}(2)$$

Now equate equations (1) and (2)

$$I^2R = \frac{I_m^2 R}{2}$$

$$I^2 = \frac{I_m^2}{2}$$

$$I = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Current $I$ is the effective value of current $i$, thus,

$$I_{eff} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Effective values for voltages are found in same way:

$$V_{eff} = \frac{V_m}{\sqrt{2}} = 0.707 V_m$$

$$E_{eff} = \frac{E_m}{\sqrt{2}} = 0.707 E_m$$

This, in words, states that

***The equivalent dc value of a sinusoidal current or voltage is*** $\frac{1}{\sqrt{2}}$ ***or 0.707 of its maximum value.***

***Example 5.9:***
Determine the effective values of

a) $i = 50\sin(wt + 20^\circ)$ mA
b) $v = 10\cos 2wt$ V

**Solution:**
Since effective values depend only on magnitude,

a) $I_{eff} = (0.707)(50) = 35.35$ m A
b) $V_{eff} = (0.707)(10) = 7.07$ V

**General equation for effective values**

The $\sqrt{2}$ relation holds only for sinusoidal waveforms. For any waveform:

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \quad \text{.................. (3)}$$

In equation 3, the integral of $i^2$ represents the area under the $i^2$ waveform. Thus,

$$I_{eff} = \sqrt{\frac{area\ under\ the\ i^2\ \text{curve}}{base}}$$

***Example 5.10:***

The 120-V dc source delivers 3.6 W to the load. Determine the peak value of the applied voltage ($V_m$) and the current ($I_m$) if the ac source is to deliver the same power to the load.

**Solution:**

$$P_{dc} = V_{dc} \times I_{dc}$$

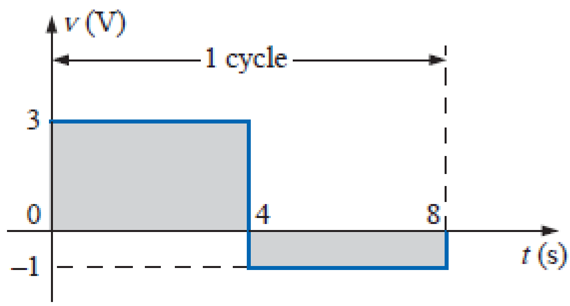
$$I_{dc} = \frac{P_{dc}}{V_{dc}} = \frac{3.6}{120} = 30 \text{ mA}$$

$$I_m = \sqrt{2} I_{dc} = 1.414 \times 30 = 42.42 \text{ mA}$$

$$V_m = \sqrt{2} V_{dc} = 1.414 \times 120 = 169.68 \text{ V}$$

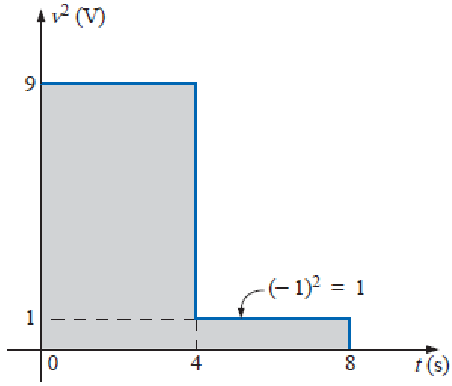
***Example 5.11:***

Find the effective or rms value of the waveform of the figure shown below:




**Solution:**

$V^2$ is shown below:




$$V_{rms} = \sqrt{\frac{(9)(4)+(1)(4)}{8}} = \sqrt{\frac{40}{8}} = 2.236 \text{ V}$$

## 5.6 Complex number review

**A complex number is a number of the form** $C = a + jb$.
Where
$a$ :is called the real part of C
b : is called its imaginary part
$j = \sqrt{-1}$
Complex numbers represent in two ways:

1. Rectangular form ( $C = a + jb$ )
2. Polar form ( $C = C\angle\theta$ )

## Conversion between Rectangular and Polar Forms

$C = a + jb$ (Rectangular form)
$C = C\angle\theta$ (Polar form)

Where C is the magnitude of C.

- **To convert from Rectangular form to Polar form**

$$C = a + jb \rightarrow C = C\angle\theta$$
$$C = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}\frac{b}{a}$$

- **To convert from Polar form to Rectangular form:**

$$C = C\angle\theta \rightarrow C = a + jb$$
$$a = C\cos\theta$$
$$b = C\sin\theta$$

***Example 5.12:***
Convert

1. from rectangular form to polar form: $C = 3 + j4$
2. from polar form to rectangular form: $C = 10\angle 30^{\circ}$

Solution:

1.

$$C = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$
$$\theta = \tan^{-1}(\frac{b}{a}) = \tan^{-1}(\frac{4}{3}) = 53.13^\circ$$
$$\therefore\ C = 5\angle 53.13^\circ$$

**2.** $a = C\cos\theta = 10\cos(30) = 10(0.866) = 8.66$
$b = C\sin = 10\sin 30^\circ = 10(0.5) = 5$
$\therefore\ C = 8.66 + j5$

## Addition and subtraction of complex number

***Example 5.13:***
Given $A = 2 + j1$ and $B = 1 + j3$. Determine their sum and difference analytically

Solution:

$A + B = (2 + j1) + (1 + j3) = 3 + j4$
$A - B = (2 + j1) - (1 + j3) = 1 - j2$

## Multiplication and Division of Complex Numbers

For multiplication, multiply magnitudes and add angles algebraically.
Thus, given $A = A\angle\theta_A$ and $B = B\angle\theta_B$

$$A.B = AB\angle\theta_A + \theta_B$$

For division, divide the magnitude of the denominator into the magnitude of the numerator, then subtract algebraically the angle of the denominator from that of the numerator.

$$A/B = A/B\angle\theta_A - \theta_B$$

***Example 5.14:***
Given $A = 3\angle 35^\circ$ and $B = 2\angle -20^\circ$, determine $A.B$ and $A/B$
Solution:
$A.B = (3\angle 35^\circ)(2\angle -20^\circ) = 6\angle 15^\circ$

$$A/B = \frac{(3\angle 35^\circ)}{(2\angle -20^\circ)} = 1.5\angle 55^\circ$$