According to Fourier series analysis, the coefficients of the Fourier series expansion of a periodic signal *x(t)* in a complex form is

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \qquad -\infty < k < \infty,$$

where *k* is the number of harmonics corresponding to the harmonic frequency of kf_0 and $\omega_0 = 2\pi / T_0$ and $f_0 = 1/T_0$ are the fundamental frequency in radians per second and the fundamental frequency in Hz, respectively.

To apply above Equation, we substitute $T_0 = NT$, $\omega_0 = 2\pi/T_0$ and approximate the integration over one period using a summation by substituting dt = T and t = nT. We obtain

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \qquad -\infty < k < \infty.$$

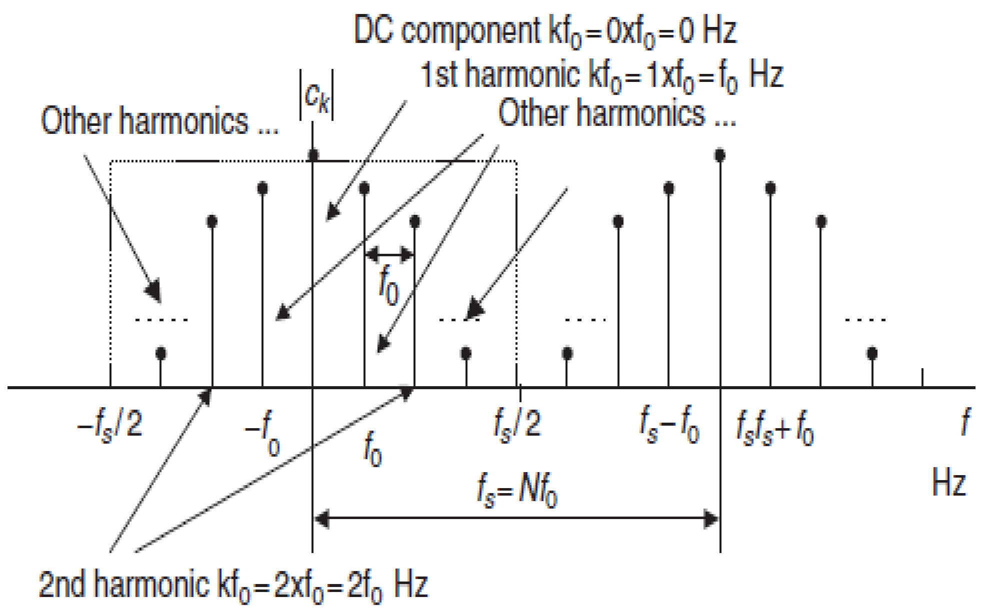
Since the coefficients c_k are obtained from the Fourier series expansion in the complex form, the resultant spectrum c_k will have two sides. There is an important feature of above Equation in which the Fourier series coefficient c_k is periodic of N. We can verify this as follows

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}e^{-j2\pi n}.$$

Since $e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$, it follows that

$$c_{k+N} = c_k.$$

Therefore, the two-sided line amplitude spectrum is periodic, as shown in Figure below.



For convenience, we compute the spectrum over the range from 0 to f_s Hz with nonnegative indices, that is,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \; k = 0,1,\ldots, N-1.$$

Example The periodic signal

$$x(t) = \sin(2\pi t)$$

is sampled using the rate f_s = 4 Hz.

Compute the spectrum c_k using the samples in one period.

Solution:

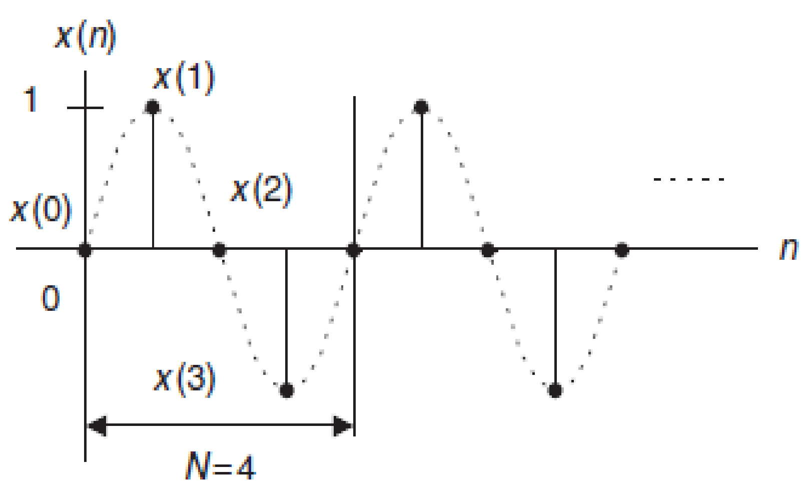
a. From the analog signal, we can determine the fundamental frequency $\omega_0 = 2\pi f_0$ radians per second and

$$f_0 = \frac{\omega_0}{2\pi} = \frac{2\pi}{2\pi} = 1Hz$$

and the fundamental period T_0 = 1 second. Since using the sampling interval $T = 1/f_s = 0.25$ second, we get the sampled signal as

$$x(n) = x(nT) = \sin(2\pi nT) = \sin(0.5\pi n)$$

and plot the first eight samples as shown in Figure



Choosing the duration of one period, N = 4, we have the sample values as follows

$$x(0) = 0;\ x(1) = 1;\ x(2) = 0;\ \text{and}\ x(3) = -1.$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0) + x(1) + x(2) + x(3)) = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi\times 1n/4} = \frac{1}{4}\left(x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\right)$$

$$= \frac{1}{4}(x(0) - jx(1) - x(2) + jx(3) = 0 - j(1) - 0 + j(-1)) = -j0.5.$$

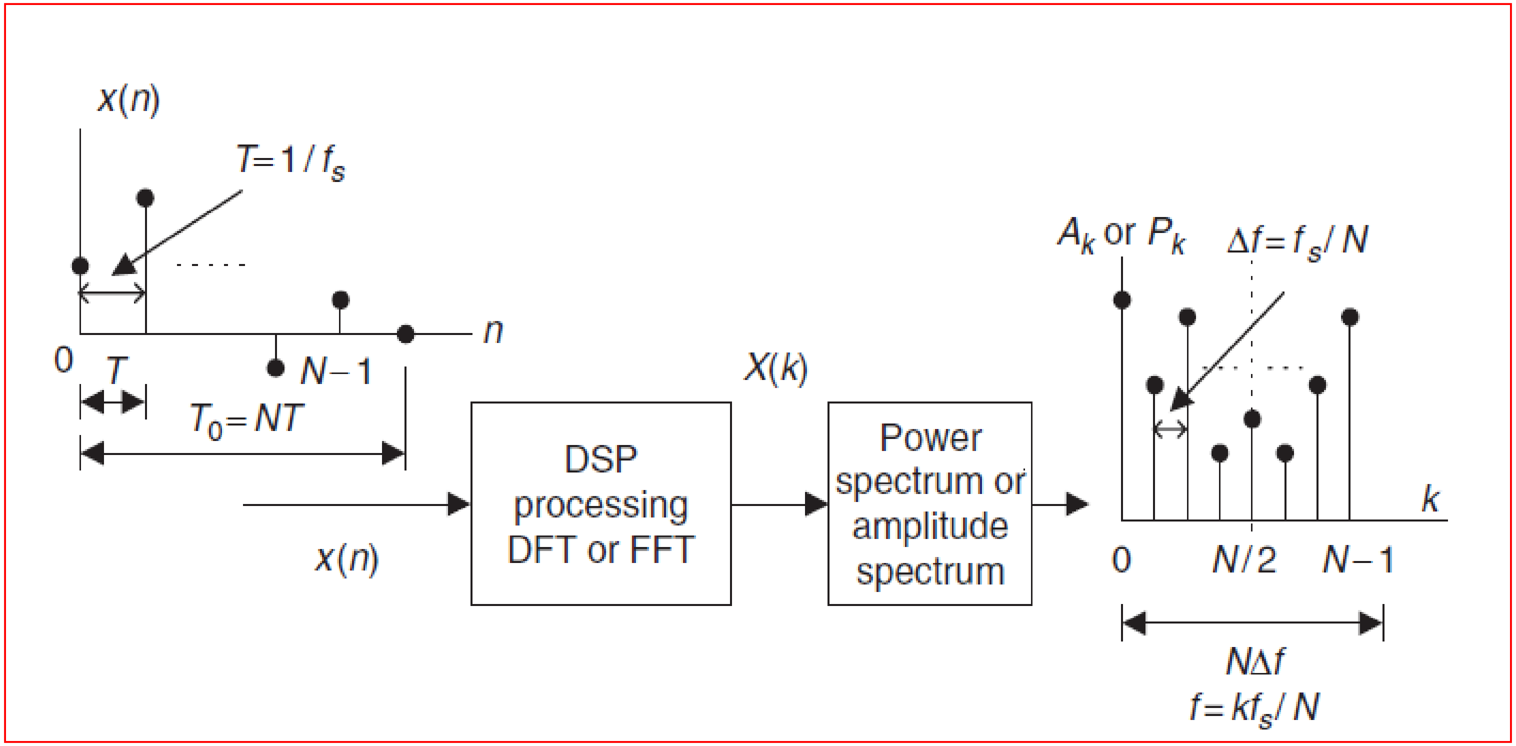
Similarly, we get

$$c_2 = \frac{1}{4}\sum_{k=0}^{3} x(n)e^{-j2\pi\times 2n/4} = 0,\ \text{and}\ c_3 = \frac{1}{4}\sum_{n=0}^{3} x(k)e^{-j2\pi\times 3n/4} = j0.5.$$

Amplitude Spectrum and Power Spectrum

One of the DFT applications is transformation of a finite-length digital signal *x(n)* into the spectrum in frequency domain.

Figure below demonstrates such an application, where A_k and P_k are the computed amplitude spectrum and the power spectrum, respectively, using the DFT coefficients *X(k)*.



$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2},$$
$$k = 0,\ 1,\ 2, \ldots,\ N-1.$$

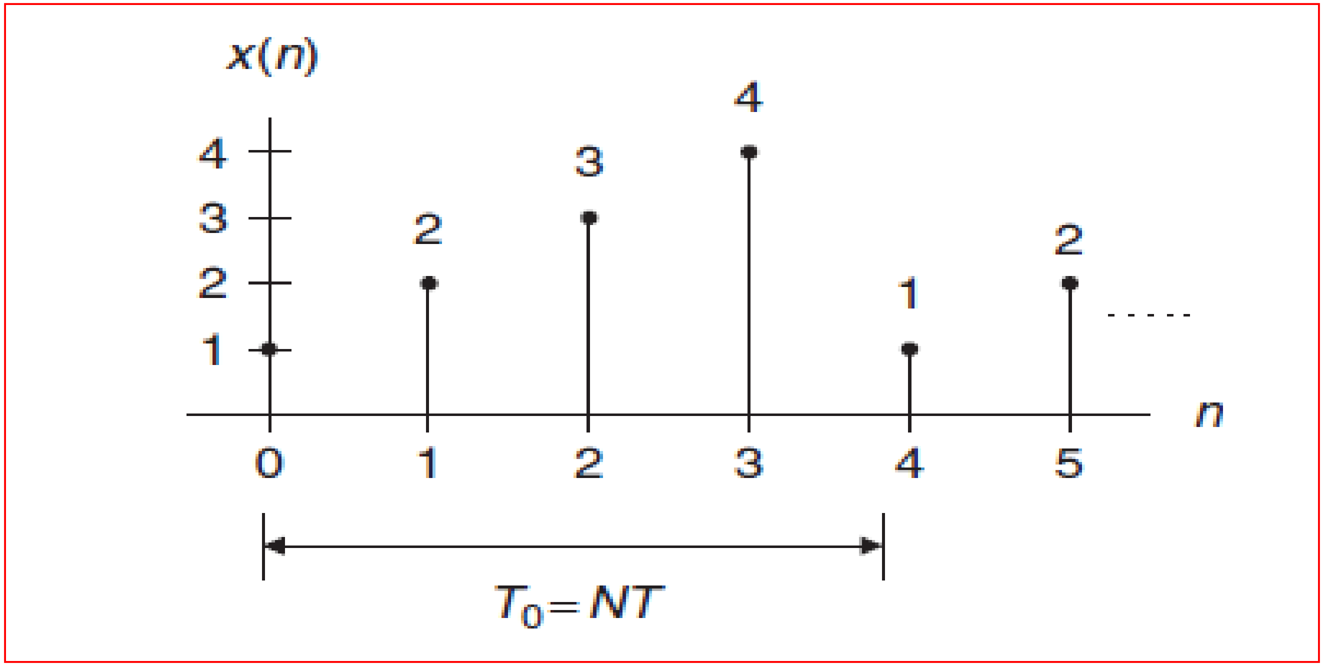
Correspondingly, the phase spectrum is given by

$$\varphi_k = \tan^{-1}\left(\frac{\text{Imag}[X(k)]}{\text{Real}[X(k)]}\right),\ k = 0,\ 1,\ 2, \ldots,\ N-1.$$

Besides the amplitude spectrum, the power spectrum is also used. The DFT power spectrum is defined as

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\left\{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2\right\},$$
$$k = 0,\ 1,\ 2, \ldots,\ N-1.$$

Example Consider the sequence



Assuming that f_s = 100 Hz, Compute the amplitude spectrum, phase spectrum, and power spectrum.

Solution: Since N = 4, and using the DFT, we find the DFT coefficients to be

$$X(0) = 10$$
$$X(1) = -2 + j2$$
$$X(2) = -2$$
$$X(3) = -2 - j2.$$

The amplitude spectrum, phase spectrum, and power density spectrum are computed as follows.

For $k = 0,\ f = k \cdot f_s/N = 0 \times 100/4 = 0\,\text{Hz},$

$$A_0 = \frac{1}{4}|X(0)| = 2.5,\ \varphi_0 = \tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}([X(0)]}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 6.25.$$

For $k = 1,\ f = 1 \times 100/4 = 25\,\text{Hz},$

$$A_1 = \frac{1}{4}|X(1)| = 0.7071,\ \varphi_1 = \tan^{-1}\left(\frac{\text{Imag}[X(1)]}{\text{Real}[X(1)]}\right) = 135^0,$$

$$P_1 = \frac{1}{4^2}|X(1)|^2 = 0.5000.$$

For $k = 2$, $f = 2 \times 100/4 = 50\,\text{Hz}$,

$$A_2 = \frac{1}{4}|X(2)| = 0.5,\ \varphi_2 = \tan^{-1}\left(\frac{\text{Imag}[X(2)]}{\text{Real}[X(2)]}\right) = 180^0,$$

$$P_2 = \frac{1}{4^2}|X(2)|^2 = 0.2500.$$

Similarly, for $k = 3$, $f = 3 \times 100/4 = 75\,\text{Hz}$,

$$A_3 = \frac{1}{4}|X(3)| = 0.7071,\ \varphi_3 = \tan^{-1}\left(\frac{\text{Imag}[X(3)]}{\text{Real}[X(3)]}\right) = -135^0,$$

$$P_3 = \frac{1}{4^2}|X(3)|^2 = 0.5000.$$