Now, we are ready to deal with the inverse z-transform using the partial fraction expansion and look-up table. The general procedure is as follows:

1. Eliminate the negative powers of z for the z-transform function X(z).

2. Determine the rational function X(z)/z (assuming it is proper), and apply the partial fraction expansion to the determined rational function X(z)/z using the formula in the following Table.

TABLE 5.3 Partial fraction(s) and formulas for constant(s).

Partial fraction with the first-order real pole:

$$\frac{R}{z-p} \qquad R = (z-p)\frac{X(z)}{z}\bigg|_{z=p}$$

Partial fraction with the first-order complex poles:

$$\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)} \qquad A = (z-P)\frac{X(z)}{z}\bigg|_{z=P}$$

P^* = complex conjugate of P

A^* = complex conjugate of A

Partial fraction with mth-order real poles:

$$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m} \qquad R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\bigg|_{z=p}$$

3. Multiply the expanded function X(z)/z by z on both sides of the equation to obtain X(z).

4. Apply the inverse z-transform using Table 5.1.

Example2 Find the inverse of the following z-transform:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

Solution: Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})}.$$

$$= \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

Then A and B are constants found using the formula in Table 5.3, that is,

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

Using Table 5.1 of the z-transform pairs, it follows that

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example3 Find *y(n)* if

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}.$$

Solution: Dividing *Y(z)* by z, we have

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}.$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}.$$

We first find B:

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4.$$

Notice that A and A*form a complex conjugate pair. We determine A as follows:

$$A = (z - 0.5 - j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$A = \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}.$$

Using the polar form, we get

$$A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

Assume that a first-order complex pole has the form

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^\circ \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^\circ.$$

We have

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}.$$

Applying the inverse z-transform from line 15 in Table 5.1 leads to

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right).$$

Using the previous formula, the inversion and subsequent simplification yield

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \phi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^\circ n - 161.57^\circ)u(n) \end{aligned}.$$

Example4

Find $x(n)$ if

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}.$$

Solution:

Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

where $A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)^2}\Big|_{z=1} = 4.$

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\Big|_{z=0.5} = \frac{-1}{(z-1)^2}\Big|_{z=0.5} = -4$$

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{z}{z-1}\Big|_{z=0.5} = -1.$$

Then $X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}.$

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n),$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n),$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n).$$

From these results, it follows that

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n).$$

Solution of Difference Equations Using the z-Transform

To solve a difference equation with initial conditions, we have to deal with time shifted sequences such as y(n _ 1), y(n _ 2), . . . , y(n _ m), and so on. Let us examine the z-transform of these terms. Using the definition of the z-transform, we have

$$Z(y(n-1)) = \sum_{n=0}^{\infty} y(n-1)z^{-n}$$
$$= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \dots$$
$$= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \dots\right)$$

It holds that

$$Z(y(n-1)) = y(-1) + z^{-1}Y(z).$$

Similarly, we can have

$$Z(y(n-2)) = \sum_{n=0}^{\infty} y(n-2)z^{-n}$$
$$= y(-2) + y(-1)z^{-1} + y(0)z^{-2} + y(1)z^{-3} + \dots$$
$$= y(-2) + y(-1)z^{-1} + z^{-2}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \dots\right)$$
$$Z(y(n-2)) = y(-2) + y(-1)z^{-1} + z^{-2}Y(z)$$
$$Z(y(n-m)) = y(-m) + y(-m+1)z^{-1} + \dots + y(-1)z^{-(m-1)}$$
$$+ z^{-m}Y(z),$$

where y(- m), y(- m + 1), . . . , y(- 1) are the initial conditions. If all initial conditions are considered to be zero, that is,

$$y(-m) = y(-m+1) = \ldots y(-1) = 0,$$

Then

$$Z(y(n-m)) = z^{-m} Y(z),$$

The following two examples serve as illustrations of applying the z-transform to find the solutions of the difference equations. The procedure is:

1. Apply z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve for the difference equation in z-transform domain.
4. Find the solution in time domain by applying the inverse z-transform.

Example A digital signal processing (DSP) system is described by the difference equation

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n).$$

Determine the solution when the initial condition is given by y(-1) = 1.

Solution: Applying the z-transform on both sides of the difference equation

$$Y(z) - 0.5\left(y(-1) + z^{-1}Y(z)\right) = 5Z(0.2^n u(n)).$$

Substituting the initial condition and $Z(0.2^n u(n)) = z/(z - 0.2)$, we achieve

$$Y(z) - 0.5\left(1 + z^{-1}Y(z)\right) = 5z/(z - 0.2).$$

Simplification yields

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z - 0.2).$$

Factoring out Y(z) and combining the right-hand side of the equation, it follows that

$$Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z - 0.2)} = \frac{z(5.5z - 0.1)}{(z - 0.5)(z - 0.2)}.$$

Then we obtain

$$Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z - 0.2)} = \frac{z(5.5z - 0.1)}{(z - 0.5)(z - 0.2)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{5.5z - 0.1}{(z - 0.5)(z - 0.2)} = \frac{A}{z - 0.5} + \frac{B}{z - 0.2},$$

$$A = (z - 0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z - 0.1}{z - 0.2}\bigg|_{z=0.5} = \frac{5.5 \times 0.5 - 0.1}{0.5 - 0.2} = 8.8333,$$

$$B = (z - 0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z - 0.1}{z - 0.5}\bigg|_{z=0.2} = \frac{5.5 \times 0.2 - 0.1}{0.2 - 0.5} = -3.3333.$$

Thus

$$Y(z) = \frac{8.8333z}{(z - 0.5)} + \frac{-3.3333z}{(z - 0.2)},$$

which gives the solution as

$$y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n).$$