Z - Transform

The z-transform is a useful tool in the analysis of discrete-time signals and systems and is the discrete-time counterpart of the Laplace transform for continuous-time signals and systems. The z-transform may be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input, and design linear filters.

The z-transform of a causal sequence *x(n)*, designated by *X(z)* or *Z(x(n))*, is defined as

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

where z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal $x(n)$ is the causal sequence, that is, $x(n) = 0$ for $n < 0$.

Thus, the definition above is referred to as a one-sided z-transform or a unilateral transform.

Z-Transform of some time sequences

1) Right Side Sequences

Example: find the Z_transform for the given sequence

$$x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$$

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{3} x(n)z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1z^{-0} + 2z^{-1} + 2z^{-2} + 1z^{-3}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

2) Left side Sequences

Example: find the Z_transform for the given sequence

$$x(n) = (1, 1, 2, \underset{\uparrow}{2})$$

$$X(z) = Z\{x(n)\} = \sum_{n=-3}^{0} x(n)z^{-n} = x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0}$$

$$= z^{3} + z^{2} + 2z + 2$$

3)Double sided sequences:

Example: find the Z_transform for the given sequence x(n)=[2 1 1 2]

$$\uparrow$$

$$X(z) = Z\{x(n)\} = \sum_{n=-2}^{1} x(n)z^{-n} = x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1}$$

$$= 2z^{2} + 1z^{1} + 1z^{0} + 2z^{-1}$$

Example Given the sequence

$$x(n) = u(n),$$

Find the z-transform of *x(n)*.

Solution: From the definition, the z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = 1 + (z^{-1}) + (z^{-1})^2 + \ldots$$

This is an infinite geometric series that converges to

$$X(z) = \frac{1}{1 - Z^{-1}}$$

$$X(z) = \frac{z}{z-1}$$

The region of convergence for all values of z is given as $|z| > 1$

Example Considering the exponential sequence

$$x(n) = a^n u(n),$$

Find the z-transform of the sequence *x(n)*.

Solution:

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + \left(az^{-1}\right) + \left(az^{-1}\right)^2 + \ldots$$

Since this is a geometric series which will converge for $|az^{-1}| < 1$ is further expressed as

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$X(z) = \frac{z}{z-a}, \text{ for } |z| > |a|.$$

TABLE 5.1 Table of z-transform pairs.

Line No.	$x(n),\ n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2-2z\cos(a)+1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2-2z\cos(a)+1}$	$\lvert z\rvert > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2-[2a\cos(b)]z+a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n \cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2-[2a\cos(b)]z+a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$
15	$2\lvert A\rvert\lvert P\rvert^n\cos(n\theta+\phi)u(n)$ where P and A are complex constants defined by $P=\lvert P\rvert\angle\theta, A=\lvert A\rvert\angle\phi$	$\frac{Az}{z-P}+\frac{A^*z}{z-P^*}$	

Example Find the z-transform for each of the following sequences:

a. $x(n) = 10u(n)$
b. $x(n) = 10\sin(0.25\pi n)u(n)$
c. $x(n) = (0.5)^n u(n)$
d. $x(n) = (0.5)^n \sin(0.25\pi n)u(n)$

Solution:

a. From Line 3 in Table 5.1, we get

$$X(z) = Z(10u(n)) = \frac{10z}{z-1}.$$

b. Line 9 in Table 5.1 leads to

$$X(z) = 10Z(\sin(0.2\pi n)u(n))$$

$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}.$$

c. From Line 6 in Table 5.1, we yield

$$X(z) = Z((0.5)^n u(n)) = \frac{z}{z - 0.5}.$$

d. From Line 11 in Table 5.1, it follows that

$$X(z) = Z((0.5)^n \sin(0.25\pi n)u(n)) = \frac{0.5 \times \sin(0.25\pi)z}{z^2 - 2 \times 0.5\cos(0.25\pi)z + 0.5^2}$$

$$= \frac{0.3536z}{z^2 - 1.4142z + 0.25}.$$

Properties of the z-Transform

Linearity: The z-transform is a linear transformation, which implies

$$Z(ax_1(n)+bx_2(n)) = aZ(x_1(n)) + bZ(x_2(n)),$$

where $x_1(n)$ and $x_2(n)$ denote the sampled sequences, while a and b are the arbitrary constants.

Example Find the z-transform of the sequence defined by

$$x(n) = u(n) - (0.5)^n u(n).$$

Solution: Applying the linearity of the z-transform previously discussed, we have

$$x(n) = u(n) - (0.5)^n u(n).$$

Using Table 5.1 yields

$$Z(u(n)) = \frac{z}{z-1}$$

and $Z(0.5^n u(n)) = \dfrac{z}{z-0.5}.$

Substituting these results into X(z) leads to the final solution,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}.$$

Shift theorem: Given *X(z)*, the z-transform of a sequence *x(n)*, the z-transform of *x(n - m),* the time-shifted sequence, is given by

$$Z(x(n-m)) = z^{-m}X(z).$$

Example Determine the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \cdot u(n-5),$$

Using Table 5.1 leads to

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}.$$

Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k) x_2(k),$$

In z-transform domain, we have

$$X(z) = X_1(z) X_2(z).$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

Example Given two sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$
$$x_2(n) = 2\delta(n) - \delta(n-1),$$

a) Find the z-transform of their convolution:

$$X(z) = Z(x_1(n)*x_2(n)).$$

b) Determine the convolution sum using the z-transform:

$$x(n) = x_1(n)*x_2(n) = \sum_{k=0}^{\infty} x_1(k)x_2(n-k).$$

Solution:

a) Applying z-transform to $x_1(n)$ and $x_2(n)$, respectively, it follows that

$$X_1(z) = 3 + 2z^{-1}$$
$$X_2(z) = 2 - z^{-1}.$$

Using the convolution property, we have

$$X(z) = X_1(z)X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$
$$= 6 + z^{-1} - 2z^{-2}.$$

b) Applying the inverse z-transform and using the shift theorem and line 1 of Table 5.1 leads to

$$x(n) = Z^{-1}\left(6 + z^{-1} - 2z^{-2}\right) = 6\delta(n) + \delta(n-1) - 2\delta(n-2).$$

Inverse z-Transform

The z-transform of the sequence x(n) and the inverse z-transform of the function X(z) are defined as, respectively,

$$X(z) = Z(x(n))$$

$$\text{and } x(n) = Z^{-1}(X(z)),$$

Where Z( ) is the z-transform operator, while Z^{-1}( ) is the inverse z-transform operator.

The inverse z-transform may be obtained by at least three methods:

- Partial fraction expansion and look-up table.
- Power series expansion.
- Residue method.

Example1 Find the inverse z-transform for each of the following functions:

a. $X(z) = 2 + \frac{4z}{z-1} - \frac{z}{z-0.5}$

b. $X(z) = \frac{5z}{(z-1)^2} - \frac{2z}{(z-0.5)^2}$

c. $X(z) = \frac{10z}{z^2 - z + 1}$

d. $X(z) = \frac{z^{-4}}{z-1} + z^{-6} + \frac{z^{-3}}{z+0.5}$

Solution:

a. $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-0.5}\right).$

From Table 5.1, we have

$x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n).$

b. $x(n) = Z^{-1}\left(\frac{5z}{(z-1)^2}\right) - Z^{-1}\left(\frac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\frac{z}{(z-1)^2}\right) - \frac{2}{0.5}Z^{-1}\left(\frac{0.5z}{(z-0.5)^2}\right).$

Then $x(n) = 5nu(n) - 4n(0.5)^n u(n).$

c. Since $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$,

by coefficient matching, we have

$$-2\cos(a) = -1.$$

Hence, $\cos(a) = 0.5$, and $a = 60°$. Substituting $a = 60°$ into the sine function leads to

$$\sin(a) = \sin(60°) = 0.866.$$

Finally, we have

$$x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n \cdot 60^0)$$

$$= 11.547\sin(n \cdot 60^0).$$

d. Since

$$x(n) = Z^{-1}\left(z^{-5}\frac{z}{z-1}\right) + Z^{-1}\left(z^{-6} \cdot 1\right) + Z^{-1}\left(z^{-4}\frac{z}{z+0.5}\right),$$

using Table 5.1 and the shift property, we get

$$x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4).$$