Difference Equations and Impulse Responses

Format of Difference Equation

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M), \quad (1)$$

where $a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b$ are the coefficients of the difference equation. Equation (1) can further be written as

$$y(n) = -a_1 y(n-1) - \ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \quad (2)$$

Or

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j). \quad (3)$$

Notice that y(n) is the current output, which depends on the past output samples y(n _ 1), . . . , y(n _ N), the current input sample x(n), and the past input samples, x(n _ 1), . . . , x(n _ N).

We will examine the specific difference equations in the following examples.

Example

Given the following difference equation:

$$y(n) = 0.25y(n-1) + x(n),$$

Identify the nonzero system coefficients.

Solution:

Comparison with Equation (2) leads to

$$b_0 = 1$$
$$-a_1 = 0.25,$$

System Representation Using Its Impulse Response:

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input δ(n) with zero initial conditions, depicted in Figure (1).

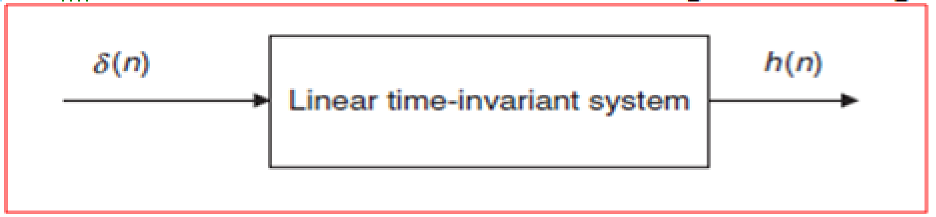


Figure (1): Unit-impulse response of the linear time-invariant system.

With the obtained unit-impulse response h(n), we can represent the linear time-invariant system in Figure (2).

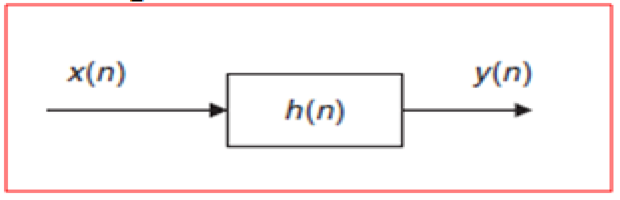


Figure (2): Representation of a linear time-invariant system using the impulse response.

Example:

Given the linear time-invariant system

$y(n) = 0.5x(n) + 0.25x(n - 1)$ with an initial condition $x(-1) = 0$,

a) Determine the unit-impulse response $h(n)$.

b) Draw the system block diagram.

c) Write the output using the obtained impulse response.

Solution:

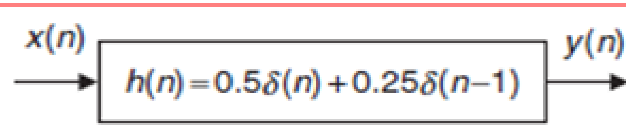
a) According to Figure 1, let $x(n) = \delta(n)$, then

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b) The block diagram of the linear time-invariant system is shown as



c) The system output can be rewritten as

$$y(n) = h(0)x(n) + h(1)x(n-1).$$

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as

$$y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots. \quad (3)$$

Equation (3) is called the digital convolution sum, which will be explored in a later section. We can verify Equation (3) by substituting the impulse sequence $x(n) = \delta(n)$ to get the impulse response

$$h(n) = \ldots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \ldots,$$

Example:

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

a) Determine the unit-impulse response $h(n)$.
b) Draw the system block diagram.
c) Write the output using the obtained impulse response.
d) For a step input $x(n) = u(n)$, verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3).

Solution:

a) Let $x(n) = \delta(n)$, then

$$h(n) = 0.25h(n-1) + \delta(n).$$

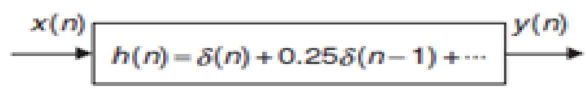
To solve for $h(n)$, we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$
$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625$$

With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots.$$

b) The system block diagram is given in Figure below.



c) The output sequence is a sum of infinite terms expressed as

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$
$$= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots$$

d) From the difference equation and using the zero-initial condition, we have

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$
$$n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1$$
$$n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25$$
$$n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125$$
$$\ldots.$$

Applying the convolution sum in Equation (3) yields

$$y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots$$

$$n = 0,\ y(0) = x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots$$
$$= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1$$

$$n = 1,\ y(1) = x(1) + 0.25x(0) + 0.0625x(-1) + \ldots$$
$$= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25$$

$$n = 2,\ y(2) = x(2) + 0.25x(1) + 0.0625x(0) + \ldots$$
$$= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125$$

Digital Convolution

Given a linear time-invariant system, we can determine its unit-impulse response *h(n)*, which relates the system input and output. To find the output sequence *y(n)* for any input sequence *x(n)*, we write the digital convolution as shown in Equation (3) as:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots \quad (4)$$

Using a conventional notation, we express the digital convolution as

$$y(n) = h(n) * x(n). \quad (5)$$

Note that for a causal system, which implies its impulse response

$$h(n) = 0 \text{ for } n < 0,$$

The lower limit of the convolution sum begins at 0 instead of 1, that is

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k).$$

We will focus on evaluating the convolution sum based on Equation (4). Let us examine first a few outputs from Equation (4):