Fast Fourier Transform

FFT is a very efficient algorithm in computing DFT coefficients and can reduce a very large amount of computational complexity (multiplications). Without loss of generality, we consider the digital sequence x(n) consisting of 2m samples, where m is a positive integer—the number of samples of the digital sequence x(n) is a power of 2, N = 2, 4, 8, 16, etc. If x(n) does not contain 2m samples, then we simply append it with zeros until the number of the appended sequence is equal to an integer of a power of 2 data points.

In this section, we focus on two formats. One is called the decimation in- frequency algorithm, while the other is the decimation-in-time algorithm. They are referred to as the radix-2 FFT algorithms.

Method of Decimation-in-Frequency

We begin with the definition of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \text{ for } k = 0, 1, \ldots, N-1,$$

The above equation can be expanded as

$$X(k) = x(0) + x(1) W_N^k + \ldots + x(N-1) W_N^{k(N-1)}.$$

Again, if we split the above equation into

$$X(k) = x(0) + x(1) W_N^k + \ldots + x\left(\frac{N}{2} - 1\right) W_N^{k(N/2-1)}$$
$$+ x\left(\frac{N}{2}\right) W^{kN/2} + \ldots + x(N-1) W_N^{k(N-1)}$$

Then we can rewrite as a sum of the following two parts

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}.$$

Modifying the second term

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}.$$

Recall

$$W_N^{N/2} = e^{-j\frac{2\pi(N/2)}{N}} = e^{-j\pi} = -1;$$

Then we have

$$X(k) = \sum_{n=0}^{(N/2)-1} \left(x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right) W_N^{kn}.$$

Now letting $k = 2m$ as an even number achieves

$$X(2m) = \sum_{n=0}^{(N/2)-1} \left(x(n) + x\left(n + \frac{N}{2}\right)\right) W_N^{2mn},$$

While substituting $k = 2m + 1$ as an odd number yields

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} \left(x(n) - x\left(n + \frac{N}{2}\right)\right) W_N^{n} W_N^{2mn}.$$

Using the fact that

$$W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2},$$

It follows that

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n) \text{ with } (N/2) \text{ points}\}$$

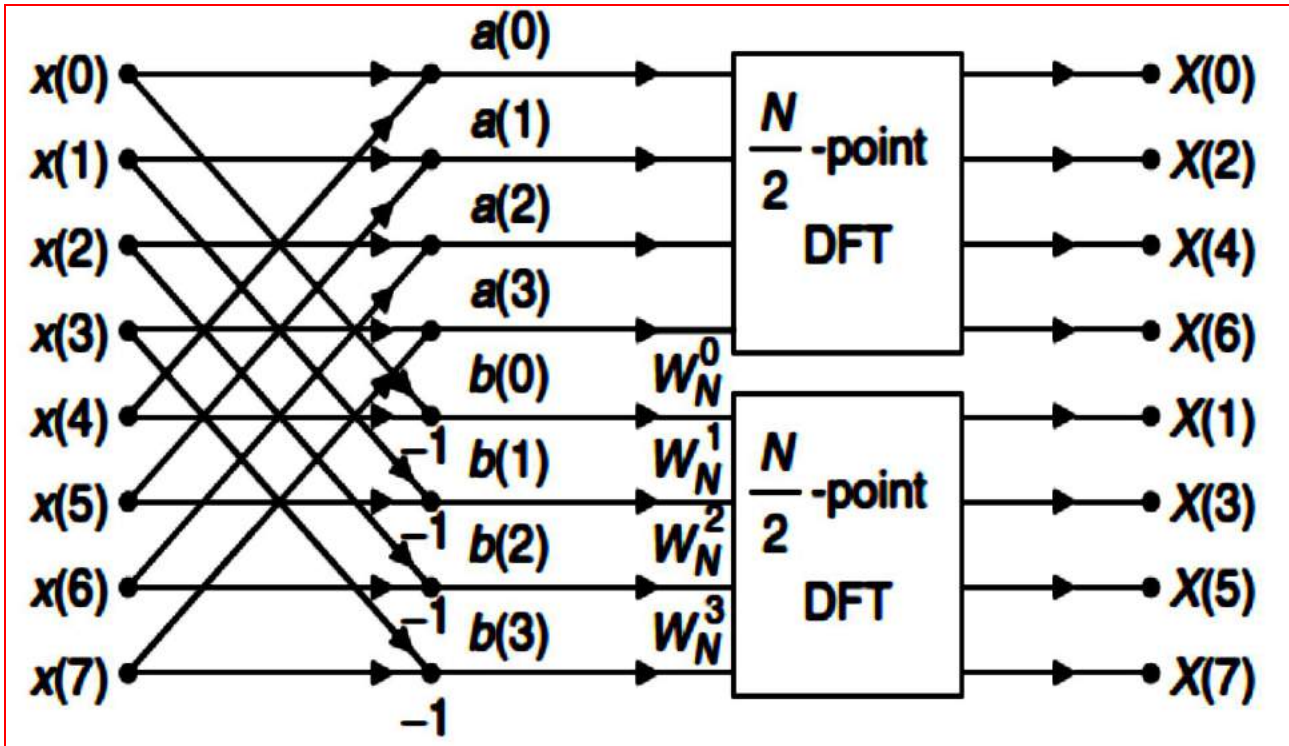
$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^{n} W_{N/2}^{mn} = DFT\{b(n) W_N^n \text{ with } (N/2) \text{ points}\},$$

Where *a(n)* and *b(n)* are introduced and expressed as

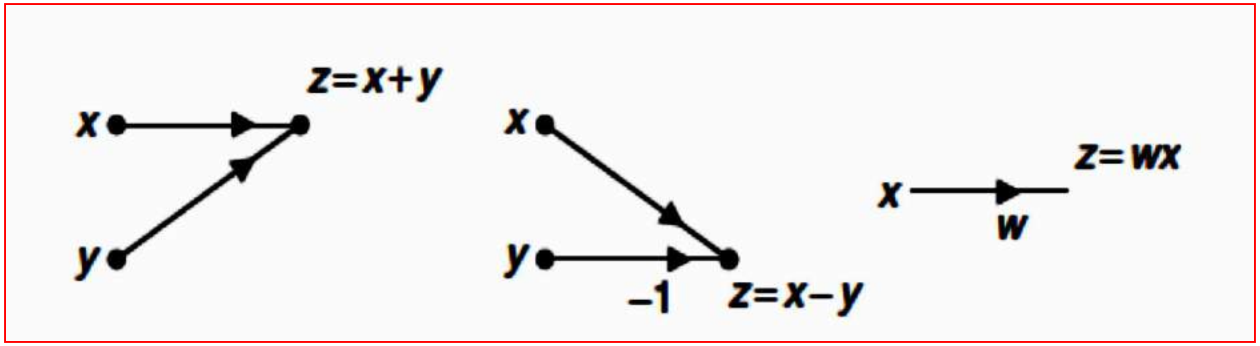
$$a(n) = x(n) + x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1\ldots,\frac{N}{2} - 1$$

$$b(n) = x(n) - x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1,\ldots,\frac{N}{2} - 1.$$

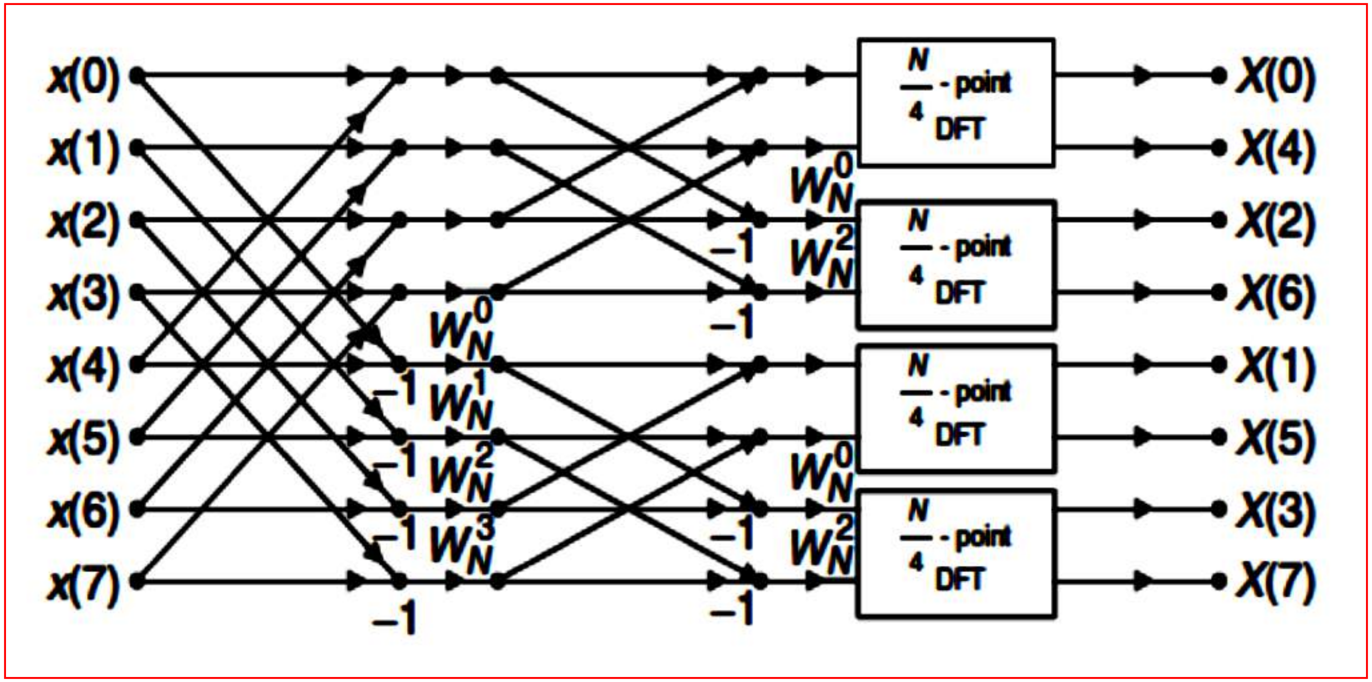
The computation process can be illustrated in the figure below

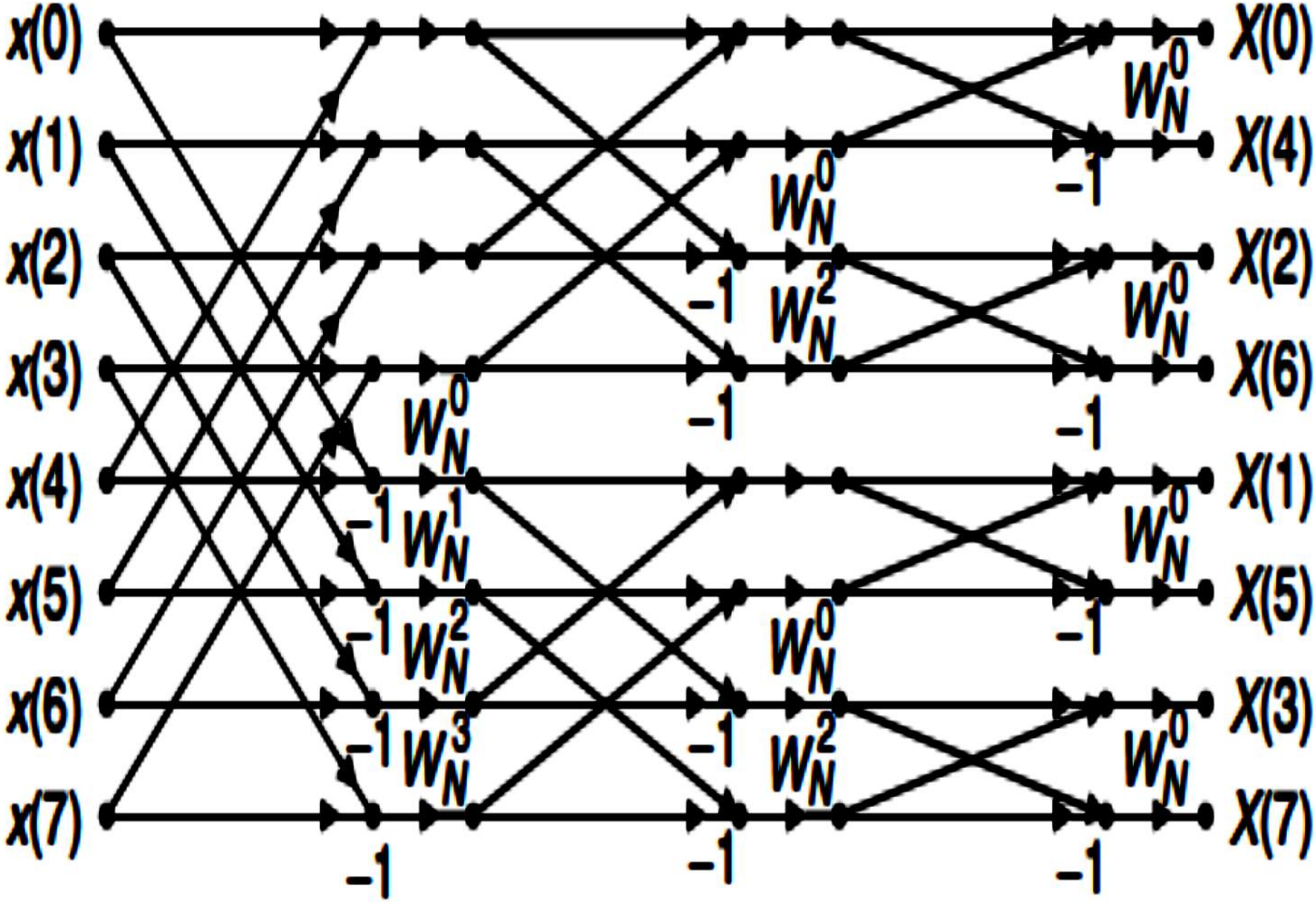


As shown in this figure, there are three graphical operations, which are illustrated in the figure below



If we continue the process described by the above figure, we obtain the block diagrams shown in following two figures.




x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
W_N^0
W_N^1
W_N^2
W_N^3
W_N^0
W_N^2
W_N^0
W_N^2
W_N^0
W_N^0
W_N^0
W_N^0
−1
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

$$\text{Complex multiplications of DFT} = N^2, \text{ and}$$

$$\text{Complex multiplications of FFT} = \frac{N}{2}\log_2(N).$$

Next, the index (bin number) of the eight-point DFT coefficient X(k) becomes 0, 4, 2, 6, 1, 5, 3, and 7, respectively, which are not in the natural order. This can be fixed by index matching. Index matching between the input sequence and the output frequency bin number by applying reversal bits is described in table shown below

Input Data	Index Bits	Reversal Bits	Output Data
$x(0)$	000	000	$X(0)$
$x(1)$	001	100	$X(4)$
$x(2)$	010	010	$X(2)$
$x(3)$	011	110	$X(6)$
$x(4)$	100	001	$X(1)$
$x(5)$	101	101	$X(5)$
$x(6)$	110	011	$X(3)$
$x(7)$	111	111	$X(7)$

Example

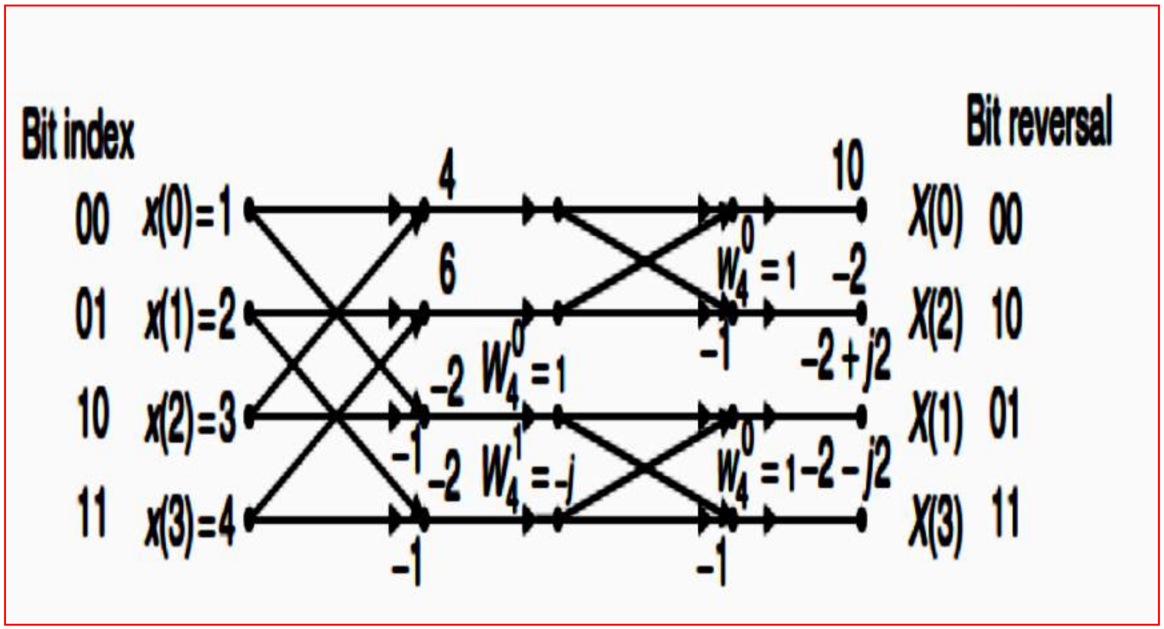
Given a sequence *x(n)* for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

a. Evaluate its DFT *X(k)* using the decimation-in-frequency FFT method.

b. Determine the number of complex multiplications.

Solution:

a. Using the FFT block diagram shown in the figure below



$$\frac{N}{2}\log_2(N) = \frac{4}{2}\log_2(4) = 4.$$